Homework: elementary particle dynamics
possible after epd1

2.1. Calculate the ratio of the gravitational attraction to the electrical repulsion between two stationary electrons. (Do I need to tell you how far apart they are?)

2.2. Sketch the lowest-order Feynman diagram representing *Delbruck scattering*: $\gamma+\gamma \to \gamma+\gamma$. (This process, the scattering of light by light, has no analog in classical electrodynamics.)

2.3. Draw all the fourth-order (four vertex) diagrams for Compton scattering $\gamma + e^- \to \gamma + e^-$. (There are 17 of them; disconnected diagrams don't count.)

possible after epd3

2.6. Draw all the lowest-order diagrams contributing to the process $e^+ + e^- \to W^+ + W^-$. (One of them involves the direct coupling of Z to W's and another the coupling of γ to W's, so when LEP (the electron-positron collider at CERN) achieved sufficient energy to make two W's, in 1996, these exotic processes could be studied experimentally. See B. Schwarzschild, *Physics Today* (September 1996), p. 21.

2.8. Some decays involve two (or even all three) different forces. Draw possible Feynman diagrams for the following processes:

(a) $\mu \to e + e + e^+ + \nu_\mu + \bar{\nu}_e$

What interactions are involved? (Both these decays have been observed, by the way.)

possible after epd4

2.5. (a) Which decay do you think would be more likely,

$$\Xi^- \to \Lambda + \pi^- \qquad \text{or} \qquad \Xi^- \to n + \pi^-$$

Explain your answer, and confirm it by looking up the experimental data.

(b) Which decay of the $D^0(c\bar{u})$ meson is most likely,

$$D^0 \to K^- + \pi^+ \ , \quad D^0 \to \pi^- + \pi^+ \ , \text{or} \quad D^0 \to K^+ + \pi^-$$

Which is *least* likely? Draw the Feynman diagrams, explain your answer and check the experimental data.
(One of the successful predictions of the Cabibbo/GIM/KM model was that charmed mesons should decay preferentially into strange mesons, even though *energetically* the 2π mode is favored.)

2.8. Some decays involve two (or even all three) different forces. Draw possible Feynman diagrams for the following processes:

(b) $\Sigma^+ \to p + \gamma$

What interactions are involved? (Both these decays have been observed, by the way.)

2.12. The W^- was discovered in 1983 at CERN, using proton/antiproton scattering:

$$p + \bar{p} \to W^- + X$$

where X represents one or more particles. What is the most likely X, for this process? Draw a Feynman diagram for your reaction. and explain why your X is more probable than the various alternatives.

possible after sr2

2.4. Determine the mass of the virtual photon in each of the lowest-order diagrams for Bhabha scattering $e^+ + e^- \to e^+ + e^-$ (assume the electron and positron are at rest). What is the photons velocity? (Note that these answers are impossible for *real* photons.)

Homework: relativistic kinematics
possible after sr1

3.9. Given two four-vectors, $a^\mu = (3, 4, 1, 2)$ and $b^\mu = (5, 0, 3, 4)$, find

(a) a_μ, b_μ

(b) $(\vec{a})^2$, $(\vec{b})^2$

(c) $\vec{a} \cdot \vec{b}$

(d) a^2, b^2

(e) $a \cdot b$

(f) Characterize a^μ and b^μ as timelike, spacelike, or lightlike.

3.10. A second–rank tensor is called *symmetric* if it is unchanged when you switch the indices ($s^{\nu\mu} = s^{\mu\nu}$); it is called *antisymmetric* if it changes sign ($a^{\nu\mu} = -a^{\mu\nu}$).

(a) How many independent elements are there in a symmetric tensor? (Since $s^{12} = s^{21}$, these would count as only *one* independent element.)

(b) How many independent elements are there in an antisymmetric tensor?

(c) Show that symmetry is preserved by Lorentz transformations — that is, if $s^{\mu\nu}$ is symmetric, so too is $s'^{\mu\nu}$. What about *anti*symmetry?

(d) If $s^{\mu\nu}$ is symmetric, show that $s_{\mu\nu}$ is also symmetric.
If $a^{\mu\nu}$ is antisymmetric, show that $a_{\mu\nu}$ is antisymmetric.

(e) If $s^{\mu\nu}$ is symmetric and $a^{\mu\nu}$ is antisymmetric, show that $s^{\mu\nu} a_{\mu\nu} = 0$.

(f) Show that any second-rank tensor ($t^{\mu\nu}$) can be written as the sum of an anti-symmetric part ($a^{\mu\nu}$) and a symmetric part ($s^{\mu\nu}$): ($t^{\mu\nu} = a^{\mu\nu} + s^{\mu\nu}$). Construct ($a^{\mu\nu}$) and ($s^{\mu\nu}$) explicitly, given ($t^{\mu\nu}$).

Remark: It helps to read the corresponding chapters in Griffiths …
if this is not enough for 3.10, it might help to read "Special Relativity for Particle Physics":
`http://web.vu.lt/ff/t.gajdosik/files/2014/01/sr4wop.pdf`

3.15. A pion traveling at speed v decays into a muon and a neutrino, $\pi^- \to \mu^- + \bar{\nu}$. If the neutrino emerges at 90° to the original pion direction, at what angle does the muon come off?

possible after sr2

3.3. Transformation between the frames S' and S, which are moving with the speed v relative to each other:

   (a) How do *volumes* transform?
   Specifically, if a container has volume V' in its own rest frame, S', what is its volume V as measured by an observer in S, with respect to which is is moving at speed v.

   (b) How do *densities* transform?
   If a container holds ρ' molecules per unit volume in its own rest frame, S', how many molecules per unit volume does it carry in S?

3.4. Cosmic ray muons are produced high in the atmosphere (at 8000 m, say) and travel toward the earth at very nearly the speed of light, (0.998 c, say). The speed of light is roughly $3. \times 10^8$ m/s.

   (a) Given the lifetime of the muon (2.2×10^{-6} s), how far would the average muon go before disintegrating, according to prerelativistic physics? Would the muon make it to ground level?

   (b) Now answer the same question using *relativistic* physics. (Because of time dilation, the muons last longer, so they travel farther.)

   (c) Pions are also produced in the upper atmosphere. In fact, the sequence is: a proton (from outer space) hits a proton (in the atmosphere) $\to p + p +$ pions. The pions then decay into muons: $\pi^\pm \to \mu^\pm + \nu_\mu(\bar{\nu}_\mu)$. But the lifetime of the pion is much shorter, (2.6×10^{-8} s). Assuming the pions have the same speed (0.998 c), will the average pion reach ground level?

   (*) Now analyze the same process from the perspective of the *muon*. (In *its* reference frame it only lasts 2.2×10^{-6} s; how then does it make it to the ground?)

3.5. Half the muons in a monoenergetic beam decay in the first 600 m. How fast are they going?

3.6. As the outlaws escape in their getaway car, which goes $\frac{3}{4}$ c, the cop fires a bullet from the pursuit car, which goes $\frac{1}{2}$ c. The muzzle velocity (speed relative to the gun) of the bullet is $\frac{1}{3}$ c. Does the bullet reach its target

   (a) According to prerelativistic physics?

   (b) According to relativity?

3.11. A particle is traveling at $\frac{3}{5}$ c in $\hat{x}$-direction. Determine its proper velocity $\eta^\mu = \frac{1}{m}p^\mu$ (all four components).

3.13. Is p^μ timelike, spacelike, or lightlike for a (real) particle with mass m?
How about a massless particle?
How about a virtual particle?

3.23. A particle travelling at speed u approaches an identical particle at rest.

   (a) What is the speed v of each particle in the CM frame?
   (*Classically*, of course, it would just be $u/2$.)

(b) Find $\gamma \equiv 1/\sqrt{1-v^2/c^2}$ in terms of $\gamma' \equiv 1/\sqrt{1-u^2/c^2}$.

(c) Use your result in part (b) to express the kinetic energy of each particle in the CM frame, and thus re-derive eq. (3.54).

possible after sr3

3.16. Particle A (energy E) hits particle B (at rest), producing particles C_1, C_2, $\ldots C_n$. Calculate the threshold (i.e. the minimum E) for this reaction, in terms of the various particle masses.

3.17. Use the result of Problem 3.16 to find the threshold energies for the following reactions, assuming that the target proton is stationary:

(a) $p+p \to p+p+\pi^0$

(b) $p+p \to p+p+\pi^+ + \pi^-$

(c) $\pi^- + p \to p + \bar{p} + n$

(d) $\pi^- + p \to K^0 + \Sigma^0$

(e) $p+p \to p+\Sigma^+ + K^0$

3.18. The first man-made Ω^- (Fig. 1.9) was created by firing a high-energy proton at a stationary hydrogen atom to produce a K^+/K^- pair: $p+p \to p+p+K^+ + K^-$; the K^- in turn hit another stationary proton, $K^- + p \to \Omega^- + K^0 + K^+$. What minimum kinetic energy is required (for the incident proton), to make an Ω^- in this way? (Gell-Mann must have done this calculation to see whether the experiment would be feasible.)

3.19. Particle A, at rest, decays into particles B and C $(A \to B + C)$.

(a) Find the energy of the outgoing particles in terms of the various masses.

(b) Find the magnitude of the outgoing momenta.

(c) Note that the triangle function $\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$ factors: $\lambda(a^2,b^2,c^2) = (a+b+c)(a-b+c)(a+b-c)(a-b-c)$. Thus $|\vec{p}_B|$ goes to zero when $m_A = m_B + m_C$, and runs imaginary when $m_A < (m_B + m_C)$. Explain.

3.20. Use the result of Problem 3.19 to find the CM energy of each decay product in the following reactions:

(a) $\pi^- \to \mu^- + \bar{\nu}$

(c) $\pi^0 \to \gamma + \gamma$

(c) $K^+ \to \pi^+ + \pi^0$

(d) $\Lambda \to p + \pi^-$

(e) $\Omega^- \to \Lambda + K^-$

3.21. A pion at rest decays into a muon and a neutrino $(\pi^- \to \mu^- + \bar{\nu}_\mu)$. On the average, how far will the muon travel (in vacuum) before disintegrating?

(*) The length of a muon track is measured to be about 0.6 mm. How do you explain this?

3.22. Particle A, at rest, decays into three or more particles: $A \to B + C + D + \dots$.

(a) Determine the maximum and the minimum energies that particle B can have in such a decay, in terms of the various masses $(m_A, m_B, m_C, m_D, \dots)$.

(b) Find the maximum and minimum electron energies in muon decay, $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$.

3.25. In a two–body scattering event, $(A + B \to C + D)$, it is convenient to introduce the *Mandelstam variables*

$$s = (p_A + p_B)^2 \qquad t = (p_A - p_C)^2 \qquad u = (p_A - p_D)^2$$

(a) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$.

The *theoretical* virtue of the Mandelstam variables is that they are Lorentz invariants, with the same value in any inertial system. *Experimentally*, though, the more accessible parameters are energies and scattering angles.

(b) Find the CM energy of A, in terms of s, t, u, and the masses.

(c) Find the Lab (B at rest) energy of A.

(d) Find the total CM energy ($E_{\mathrm{TOT}} = E_A + E_B = E_C + E_D$).

3.27. Work out the kinematics of Compton scattering: a photon of wavelength λ collides elastically with a charged particle of mass m. If the photon scatters at angle θ, find its outgoing wavelength, λ'; use $E_\gamma = h/\lambda$.

possible after sr4

3.7. Find the matrix $\mathbf{M}$ that inverts the Lorentz transformation Λ, eq. (3.12): $x^\mu = M^\mu_\nu x^{\nu\prime}$. Show that $\mathbf{M}$ is the matrix inverse of Λ: $\Lambda\mathbf{M} = 1$.

3.8. Show that $\mathbf{I} = x_\mu x^\mu$, eq. (3.13), is invariant under the Lorentz transformation Λ, eq. (3.8).

3.12. Consider a collision in which particle A (with 4-momentum p_A^μ) hits particle B (4-momentum p_B^μ), producing particles C (p_C^μ) and D (p_D^μ). Assume energy-momentum conservation in system S (i.e. $p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu$). Using the Lorentz transformation $\Lambda(\eta)$, eq. (3.12), show that energy and momentum are also conserved in system S'.

exercises from David Hogg, Chapter 6, p. 34
possible after sr3

6.7. A particle of mass M, at rest, decays into two smaller particles of masses m_1 and m_2. What are their energies and momenta?

6.8. Solve problem 6.7 again for the case $m_2 = 0$. Solve the equations for p and E_1 and then take the limit $m_1 \to 0$.

6.9. If a massive particle decays into photons, explain using 4-momenta why it cannot decay into a single photon, but must decay into two or more. Does your explanation still hold if the particle is moving at high speed when it decays?

6.10. A particle of rest mass M, travelling at speed v in the x-direction, decays into two photons, moving in the positive and negative x-direction relative to the original particle. What are their energies? What are the photon energies and directions if the photons are emitted in the positive and negative y-direction relative to the original particle (i.e., perpendicular to the direction of motion, in the particles rest frame).

Homework: Symmetries — due 2018/02/27, 17:00

David Griffiths, Chapter 4, pp. 137-138, n. 4.1, n. 4.2, n. 4.6, and n. 4.7:

4.1. Prove that I, R_+, R_-, R_a, R_b, and R_c are *all* the symmetries of the equilateral triangle.

4.2. Construct a "multiplication table" for the triangle group.
Is this an Abelian group? How can you tell, just by looking at the multiplication table?

4.6. Consider a vector $\vec{a}$ in two dimensions. Suppose its components with respect to Cartesian axes x, y, are (a_x, a_y). What are its components (a'_x, a'_y) in a system x', y' which is rotated, counterclockwise, by an angle θ, with respect to x, y? Express your answer in the for of a 2×2 matrix $R(\theta)$:

$$\begin{pmatrix} a'_x \\ a'_y \end{pmatrix} = R(\theta) \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

Show that R is an orthogonal matrix. What is its determinant? The set of *all* such rotations constitutes a group; what is the name of this group? By multiplying the matrices show that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$; is this an Abelian group?

4.7. Consider the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Is it in the group $O(2)$? How about $SO(2)$? What is its effect on the vector $\vec{a}$ of Problem 4.6? Does it describe a possible rotation of the plane?

4.8. Suppose we interpret the electron literally as a classical solid sphere of radius r, mass m, spinning with angular momentum $\frac{1}{2}\hbar$. What is the speed, v, of a point on its 'equator'? Experimentally, it is known that r is less than 10^{-16} cm. What is the corresponding equatorial speed? What do you conclude from this?

4.19. (a) Show that $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$. ('1' here really means the 2×2 unit matrix; if no matrix is specified, the unit matrix is understood.)

(b) Show that $\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$; $\sigma_y\sigma_z = -\sigma_z\sigma_y = i\sigma_x$; $\sigma_z\sigma_x = -\sigma_x\sigma_z = i\sigma_y$.

These results are neatly summarized in the formula

$$\sigma_j\sigma_k = \delta_{jk} + i\epsilon_{jk\ell}\sigma_\ell$$

(summation over ℓ implied), where δ_{jk} is the Kronecker delta:

$$\delta_{jk} = \left\{ \begin{array}{ll} 1, & \text{if } j = k \\ 0, & \text{otherwise} \end{array} \right\}$$

and $\epsilon_{jk\ell}$ is the Levi-Civita symbol:

$$\epsilon_{jk\ell} = \left\{ \begin{array}{rl} 1, & \text{if } jk\ell = 123,\ 231,\ \text{or } 312 \\ -1, & \text{if } jk\ell = 132,\ 213,\ \text{or } 321 \\ 0, & \text{otherwise} \end{array} \right\}$$

4.22. (a) Show that $U(\vec{\theta}) = e^{-i(\vec{\theta}\cdot\vec{\sigma})/2}$, in eq. (4.28), is *unitary*.

(b) Show that $\det U = 1$. [*Hint*: You can either do this directly (however, see footnote after eq. (4.29)), or else use the result of Problem 4.21.]

4.23. The extension of everything in Section 4.4 to higher spin is relatively straightforward. For spin 1 we have three state ($m_s = +1,\ 0,\ -1$), which can we may represent as column vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively. The only problem is to construct the 3×3 matrices $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$. The latter is easy:

(a) Construct $\hat{S}_z$ for spin 1.

To obtain $\hat{S}_x$ and $\hat{S}_y$ it is easiest to start with the "raising" and "lowering" operators, $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$, which have the property

$$\hat{S}_\pm |sm\rangle = \hbar\sqrt{s(s+1) - m(m \pm 1)}\,|s(m \pm 1)\rangle$$

(b) Construct the matrices $\hat{S}_+$ and $\hat{S}_-$ for spin 1.

(c) Using (b) determine the spin-1 matrices $\hat{S}_x$ and $\hat{S}_y$.

(d) Do the same construction for spin $\frac{3}{2}$.