Important concepts and formulas

of Aitchison and Hay

''Gauge Field Theories in Particle Physics''

1. Quarks and Leptons

Introduction

- how to come to leptons and quarks historically
- Table 1: Properties of Quarks and Leptons, p. 27

Leptons			Quarks		
Particle	Mass (MeV/c^2)	Q/e	Particle	Mass (GeV/c^2)	Q/e
ν_e	$< 3 \times 10^{-6}$	0	u	$1 - 5 \times 10^{-3}$	$2/3$
e^-	0.511	-1	d	$3 - 9 \times 10^{-3}$	$-1/3$
ν_μ	< 0.19	0	c	$1.15 - 1.35$	$2/3$
μ^-	105.66	-1	s	$75 - 170 \times 10^{-3}$	$-1/3$
ν_τ	< 18.2	0	t	174.3 ± 5.1	$2/3$
τ^-	1777.0	-1	b	$4 - 4.4$	$-1/3$

2. Particle Interactions in the Standard Model

The Yukawa potential $$U(r) = -\frac{g_s^2}{4\pi}\frac{e^{-r/a}}{r} \tag{1}$$

- can be understood as coming from the exchange of a massive particle

$$m_U^2 c^4 \simeq \frac{c^2\hbar^2}{a^2} \quad \text{or} \quad m_U \simeq \frac{\hbar}{ac} \tag{2}$$

- the Fourier transform of $U(r)$ gives

$$f(\vec{q}) = \int d^3\vec{r}\, e^{i(\vec{q}\cdot\vec{r})} U(r) = -\frac{g_s^2}{|\vec{q}|^2 + m_U^2} \tag{3}$$

- Electromagnetic interactions: range $a \to \infty$ or $m_U \to 0$

- Weak interactions: $m_W \simeq 80\,\text{GeV}$ $\to$ range $a \ll$ fm
  $\Rightarrow$ ''point-like'' interaction, but similar coupling strength as EM

- Strong interactions: $m_g \to 0$ from scattering angles . . .

3. Electromagnetism as a Gauge Theory

Fourvector notation $A^\mu = (V, \vec{A})$

- (Four-)current $j^\mu = (\rho, \vec{j})$ with current conservation $\partial_\mu j^\mu = 0$
- Maxwells equations

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu \tag{4}$$

- the gauge transformations

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi \tag{5}$$

  leaves the fieldstrength $F^{\mu\nu}$ (and Maxwells equations) invariant

- the covariant derivative (3.29)

$$D_\mu = \partial_\mu + iqA_\mu \tag{6}$$

- gauge transformations also for electrons (3.40)

$$\Psi \to \Psi' = e^{iq\chi}\Psi \tag{7}$$

4. Relativistic Quantum Mechanics

$$E \simeq i\frac{\partial}{\partial t} \ , \vec{p} \simeq -i\frac{\partial}{\partial \vec{x}} \ , \text{ or } \quad p_\mu \simeq i\frac{\partial}{\partial x^\mu} = i\partial_\mu \tag{8}$$

- mass-shell condition

$$p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2 \tag{9}$$

- solution to the Klein-Gordon equation

$$(\Box + m^2)\phi = (\partial_\mu \partial^\mu + m^2)\phi = 0 \tag{10}$$

as plain waves: $\phi = Ne^{-ip.x} = Ne^{-iEt + i\vec{p}\cdot\vec{x}}$

- Dirac equation as the "square root" of the KG equation:

$$(i\frac{\partial}{\partial t} + i\vec{\alpha}\cdot\vec{\nabla} - \beta m)\Psi = 0 \tag{11}$$

or more covarant looking (with $\beta^2 = \mathbf{1}_{4\times 4}$)

$$\beta(i\beta\frac{\partial}{\partial t} + i\beta\vec{\alpha}\cdot\vec{\nabla} - m)\Psi = \gamma^0(i\gamma^\mu\partial_\mu - m)\Psi = 0 \tag{12}$$

with the definitions $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$

4. Relativistic Quantum Mechanics

The γ-matrices

- are defined by the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} \times \mathbf{1}_{4\times 4} \tag{13}$$

- one choice is

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & -\mathbf{1}_{2\times 2} \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \tag{14}$$

- a Dirac spinor has four components

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{15}$$

and the (Weyl-)spinors ϕ and χ have two components

4. Relativistic Quantum Mechanics

Spinors

- a basis for the spinors ϕ (or χ) can be chosen as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \uparrow \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \downarrow \tag{16}$$

- Lorentz transformation are defined on (Dirac) spinors, too . . .

- the Dirac current $\quad j^\mu = \bar{\Psi}\gamma^{mu}\Psi \qquad$ with $\bar{\Psi} := \Psi^\dagger\gamma^0$ (17)

  is conserved: $\partial_\mu j^\mu = 0$

- using the gauge principle $\qquad \partial_\mu \to D_\mu$ (18)

  in the Dirac equation allows the discussion of atoms

  – splitting the Dirac spinor into
    big (ϕ, "particle") and small ($\chi \sim \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi$, "antiparticle") components
    is convenient

5. QFT I: — free scalar field

Quantum Oscillator (algebra)

- from $[\widehat{H}, \widehat{a}] = -\omega \widehat{a}$, $[\widehat{H}, \widehat{a}^\dagger] = +\omega \widehat{a}^\dagger$, and $\widehat{H}|n\rangle = E_n|n\rangle$ (19)

  follows

$$\widehat{H}(\widehat{a}^\dagger|n\rangle) = (E_n + \omega)(\widehat{a}^\dagger|n\rangle) \text{ and } \widehat{H}(\widehat{a}|n\rangle) = (E_n - \omega)(\widehat{a}|n\rangle) \quad (20)$$

- for the spectrum to be bounded from below:

  ⇒ $\exists$ a minimum $n_0 \doteq 0$ with $\widehat{a}|0\rangle = 0$

  – $|0\rangle$ is called vacuum: no exitation present

- a field is a set of quantum oscillators at each point

  ⇒ canonical quantisation

$$[\widehat{\phi}(x), \widehat{\phi}(y)] = [\widehat{\pi}(x), \widehat{\pi}(y)] = 0 \text{ and } [\widehat{\phi}(x), \widehat{\pi}(y)] = i\delta(x - y) \quad (21)$$

- Fourier expansion ⇒ wave expansion

  ⇒ creation and annihilation operators as expansion coefficients

- action principle $\delta S = 0$ ⇒ minimizing the action
  – leads to the pathintegral quantisation

Interaction picture !

- Dyson expansion $\simeq$ free Hamiltonian H_0 + interaction Hamiltonian H_{int}
  - expansion in powers of the interaction strength
- scattering matrix $S_{fi} = \langle f|\widehat{S}|i\rangle$ with $\widehat{S} = Te^{iH_{\text{int}}t}$
  - the projection of the final state onto the time-evolved initial state
  - interaction picture is crucial here!
- Wick contraction — assumption: $\widehat{\phi} \sim \widehat{a} + \widehat{a}^\dagger$
  - not more than one creation or annhiliation operator

$$\langle 0|\widehat{A}\widehat{B}\widehat{C}\widehat{D}\ldots|0\rangle = \langle\widehat{A}\widehat{B}\rangle_0\langle\widehat{C}\widehat{D}\rangle_0\langle\ldots\rangle_0 + \text{permutations} \tag{22}$$

- since we have $S_{fi} = \langle f|\widehat{S}|i\rangle = \langle T\widehat{a}_f e^{iH_{\text{int}}t}\widehat{a}_i^\dagger\rangle_0 = \sum_n T\langle\widehat{a}_f \prod_k^n \phi_k \widehat{a}_i^\dagger\rangle_0$
  - we get as the primary element the propagator

$$\langle 0|T[\widehat{\phi}(x)\widehat{\phi}(y)]|0\rangle = \langle\widehat{\phi}_x\widehat{\phi}_y\rangle_T = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik.(x-y)}}{(k^0)^2-(\vec{k})^2-m_\phi^2+i\epsilon} \tag{23}$$

6. QFT II: — interacting scalar field

plane wave expansion

$$\hat{\phi}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E}} \left[\hat{a}(k)e^{-i\,k.x} + \hat{a}^\dagger(k)e^{i\,k.x}\right] \tag{24}$$

possibilities for connecting operators in $\langle \hat{a}_f \prod_k^n \phi_k \hat{a}_i^\dagger \rangle_0$

- $\langle \hat{a}_f \hat{a}_i^\dagger \rangle_0$ gives not connected diagrams
- $\langle \hat{a}_A(p)\hat{\phi}_A(x)\rangle_0 = \frac{1}{\sqrt{2E_A}} e^{i\,p.x}$
- $\langle \hat{\phi}_A(x)\hat{a}_A^\dagger(p)\rangle_0 = (\langle \hat{a}_A(p)\hat{\phi}_A(x)\rangle_0)^\dagger = \frac{1}{\sqrt{2E_A}} e^{-i\,p.x}$
- and finally the propagator

$$\langle 0|T[\hat{\phi}_C(x)\hat{\phi}_C(y)]|0\rangle = \langle \hat{C}_x \hat{C}_y \rangle_T = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik.(x-y)}}{(k^0)^2-(\vec{k})^2-m_C^2+i\epsilon} \tag{25}$$

Matrixelements and Transitionrate

- the scattering matrix is split into non-interaction and interaction

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4(p_f - p_i)\mathcal{M}_{fi} \tag{26}$$

- the transitionrate uses only the interaction part

$$\dot{P}_{fi} = (2\pi)^4 \delta^4(p_f - p_i)|\mathcal{M}_{fi}|^2 \tag{27}$$

- the cross section is a transition rate

$$d\sigma = \frac{\text{symmetryfactor}}{\text{fluxfactor}} \int d\text{Lips} \;\; |\mathcal{M}_{fi}|^2 \tag{28}$$

  - the symmetryfactor $= 1/(n_{\text{identical}}!)$ accounts for identical final state particles
  - the fluxfactor accounts for density of initial states and probability of interaction
    * one-particle fluxfactor $= 2E$
    * two-particle fluxfactor $= 2E_A 2E_B |v_{\text{rel}}| = 4\sqrt{(p_A.p_B)^2 - (m_A m_B)^2}$

Lorentz invariant phase space

$$d\text{Lips}(s = p_i^2; p_A, p_B, \dots) = (2\pi)^4 \delta^4(p_i - \sum_{f=A,B,\dots} p_f) \frac{d^3\vec{p}_A}{(2\pi)^3 2E_A} \frac{d^3\vec{p}_B}{(2\pi)^3 2E_B} \dots \quad (29)$$

- in the center-of-momentum frame (CM-frame) $\vec{p}_i = \vec{p}_f = 0$ (30)
- for only two particles in the final state $d\text{Lips}(s; p_A, p_B)_{\text{CM}}$

$$= \frac{(2\pi)^4 \delta(\sqrt{s} - E_A - E_B)\delta^3(\vec{p}_A + \vec{p}_B) d^3\vec{p}_A d^3\vec{p}_B}{(2\pi)^3 2E_A (2\pi)^3 2E_B} = \frac{p^2\, dp d^2\Omega \delta(\sqrt{s} - E_A - E_B)}{(4\pi)^2 E_A E_B} \quad (31)$$

– with the abbreviations $p = |\vec{p}_A|$ and $W = E_A + E_B$ we get $dW = \frac{pdp}{E_A} + \frac{pdp}{E_B} = \frac{Wpdp}{E_A E_B}$

$$d\text{Lips}(s; p_A, p_B)_{\text{CM}} = \frac{d^2\Omega}{4\pi} \frac{p}{4\pi W} dW \delta(\sqrt{s} - W) \quad (32)$$

– solving $\sqrt{s} = W = E_A + E_B$ for p we get

$$\sqrt{s} - E_A = \sqrt{E_A^2 - m_A^2 + m_B^2} \quad \Rightarrow \quad 2\sqrt{s}E_A = s + m_A^2 - m_B^2 \quad (33)$$

$$p = \sqrt{E_A^2 - m_A^2} = \frac{1}{2\sqrt{s}}\sqrt{[s + m_A^2 - m_B^2]^2 - 4sm_A^2} = \frac{1}{2\sqrt{s}}\sqrt{s^2 + m_A^4 + m_B^4 - 2sm_A^2 - 2sm_B^2 - 2m_A^2 m_B^2} \quad (34)$$

– and $\frac{1}{2}$ of the fluxfactor

$$2\sqrt{(p_A.p_B)^2 - (m_A m_B)^2} = \sqrt{(2p_A.p_B)^2 - (2m_A m_B)^2} = \sqrt{[(p_A + p_B)^2 - m_A^2 - m_B^2]^2 - (2m_A m_B)^2}$$

$$= \sqrt{[s - m_A^2 - m_B^2]^2 - (2m_A m_B)^2} = \sqrt{s^2 + m_A^4 + m_B^4 - 2sm_A^2 - 2sm_B^2 - 2m_A^2 m_B^2} = 2p\sqrt{s} \quad (35)$$

– and the total cross section $\left.\frac{d\sigma}{d^2\Omega}\right|_{\text{CM}} = \frac{|\mathcal{M}_{fi}|^2}{(8\pi)^2 s}$ (36)

7. QFT III: — complex scalar, Dirac, Maxwell, EM interactions

conservation laws

- for the real scalar, the conserved quantity was the Hamiltonian $\widehat{H} = \omega(\widehat{N} + \frac{1}{2})$ itself
- only for two (or more) real scalars (or a complex scalar) one can construct different conserved quantities
  - of course, one has to have a symmetry of the Lagrangian (action)

Example: phase invariance $\widehat{\phi}' = e^{-i\alpha}\widehat{\phi}$

- we get a conserved current $\widehat{N}^{\mu}_{\phi} = i[\widehat{\phi}^{\dagger}(\partial^{\mu}\widehat{\phi}) - (\partial^{\mu}\widehat{\phi}^{\dagger})\widehat{\phi}]$ (37)
  - conserved means $0 = \partial_{\mu}\widehat{N}^{\mu}_{\phi} = \frac{\partial}{\partial t}\widehat{N}^{0}_{\phi} + \partial_j \widehat{N}^{j}_{\phi}$
  - with the definition $\widehat{N}_{\phi} := \int d^3x\, \widehat{N}^{0}_{\phi}$
  - we get $\frac{\partial}{\partial t}\widehat{N}_{\phi} = \int d^3x \frac{\partial}{\partial t}\widehat{N}^{0}_{\phi} = -\int d^3x\, \partial_j \widehat{N}^{j}_{\phi} = -\int_{\partial V} dS_j\, \widehat{N}^{j}_{\phi} \to 0$ (38)

$\Rightarrow$ conserved charge $\widehat{N}_{\phi}$ $\Rightarrow$ $[\widehat{H}, \widehat{N}_{\phi}] = 0$
  - can be checked explicitely, too

conservation laws

- using the plane wave expansion, eq.(24), for a complex field

$$\widehat{\phi}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E}} \left[\widehat{a}(k)e^{-i\,k.x} + \widehat{b}^\dagger(k)e^{i\,k.x}\right] \tag{39}$$

  - we get the operator expression for $\widehat{N}_\phi$

$$\widehat{N}_\phi = \int \frac{d^3\vec{k}}{(2\pi)^3} \left[\widehat{a}^\dagger(k)\widehat{a}(k) - \widehat{b}^\dagger(k)\widehat{b}(k)\right] \tag{40}$$

  - giving us the operator algebra for the fields themselves

$$[\widehat{N}_\phi, \phi] = -\phi \qquad [\widehat{N}_\phi, \phi^\dagger] = +\phi^\dagger \tag{41}$$

  - with exponentiating

$$\widehat{U}(\alpha) = e^{i\alpha\widehat{N}_\phi} \tag{42}$$

  we get

$$\widehat{U}(\alpha)\widehat{\phi}\,\widehat{U}^{-1}(\alpha) = e^{-i\alpha}\widehat{\phi} = \widehat{\phi}' \tag{43}$$

7. QFT III: — complex scalar, Dirac, Maxwell, EM interactions

conservation laws

- with the counting of states $\hat{N}_\phi|n_\phi\rangle = n_\phi|n_\phi\rangle$ (44)

  and the operator algebra of the fields we get

$$\hat{N}_\phi(\phi|n_\phi\rangle) = (n_\phi - 1)(\phi|n_\phi\rangle) \qquad \hat{N}_\phi(\phi^\dagger|n_\phi\rangle) = (n_\phi + 1)(\phi^\dagger|n_\phi\rangle) \tag{45}$$

$\Rightarrow$ ϕ destroys a particle or creates an antiparticle

- definition of the vacuum: $a|0\rangle = b|0\rangle = 0$ (46)

- the propagator is now

$$\langle 0|T[\hat{\phi}(x)\hat{\phi}^\dagger(y)]|0\rangle = \langle 0|T[\hat{\phi}^\dagger(x)\hat{\phi}(y)]|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik.(x-y)}}{k^2-M^2+i\epsilon} \tag{47}$$

  - ϕ and $\phi^\dagger$ have to have the same mass

Dirac fields

- the Dirac Lagrangian is $\mathcal{L}_D = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi$ (48)

  with $\partial\!\!\!/ := \gamma^\mu \partial_\mu$ and γ^μ defined by the Clifford algebra, eq.(13)

- the Dirac fields Ψ are still superpositions of plande waves:

$$\hat{\Psi}_\alpha(x) = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E}} \sum_{s=\pm\frac{1}{2}} \left[\hat{c}_s(k) u_\alpha(k,s) e^{-i\,k.x} + \hat{d}^\dagger_s(k) v_\alpha(k,s) e^{i\,k.x}\right] \qquad (49)$$

  - $u(k,s)$ and $v(k,s)$ are commuting spinor wave functions
  - $\hat{\Psi}$ anticommute because of the anticommutation of $\hat{c}$ and $\hat{d}$:

$$\{\hat{c}_{rk}, \hat{c}_{sp}\} = \{\hat{c}^\dagger_{rk}, \hat{c}^\dagger_{sp}\} = \{\hat{d}_{rk}, \hat{d}_{sp}\} = \{\hat{d}^\dagger_{rk}, \hat{d}^\dagger_{sp}\} = \{\hat{c}^{(\dagger)}_{rk}, \hat{d}^{(\dagger)}_{sp}\} = 0 \qquad (50)$$

$$\text{and} \quad \{\hat{c}_r(k), \hat{c}^\dagger_s(p)\} = \{\hat{d}_r(k), \hat{d}^\dagger_s(p)\} = (2\pi)^3 \delta^3(\vec{k} - \vec{p})\delta_{rs} \qquad (51)$$

- $u(k,s)$ and $v(k,s)$ fulfill the completeness relations

$$\sum_{s=\pm\frac{1}{2}} u_\alpha(k,s)\bar{u}_\beta(k,s) = (k\!\!\!/ + m)_{\alpha\beta} \qquad \sum_{s=\pm\frac{1}{2}} v_\alpha(k,s)\bar{v}_\beta(k,s) = (k\!\!\!/ - m)_{\alpha\beta} \qquad (52)$$

Dirac fields

- canonical momentum $\hat{\pi} = \frac{\partial \mathcal{L}_D}{\partial \dot{\hat{\Psi}}} = \frac{\partial \mathcal{L}_D}{\partial(\partial_t \hat{\Psi})} = \bar{\Psi} i\gamma^0 = i\Psi^\dagger$ (53)

  gives the canonical commutation relations

$$\{\hat{\Psi}_\alpha(x), \hat{\Psi}^\dagger_\beta(y)\} = \delta^3(\vec{x} - \vec{y})\delta_{\alpha\beta} \tag{54}$$

- the phase transformation $\Psi' = e^{-i\alpha}\Psi \sim \Psi - i\alpha\Psi$
  - gives the conserved current $\hat{N}^\mu_\psi = \bar{\Psi}\gamma^\mu\Psi$
  - and the number operator $\hat{N}_\psi = \int d^3x \Psi^\dagger\Psi$

- in terms of creation and annihilation operators we get

$$\hat{H}_D = \int \frac{d^3\vec{k}}{(2\pi)^3} E_k \sum_{s=\pm\frac{1}{2}} \left[\hat{c}^\dagger_s(k)\hat{c}_s(k) + \hat{d}^\dagger_s(k)\hat{d}_s(k)\right] \tag{55}$$

and

$$\hat{N}_\psi = \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_{s=\pm\frac{1}{2}} \left[\hat{c}^\dagger_s(k)\hat{c}_s(k) - \hat{d}^\dagger_s(k)\hat{d}_s(k)\right] \tag{56}$$

Dirac fields

- the momentum space (i.e. Fourier transformed) Dirac propagator

$$\int d^4x\, e^{ik.x} \langle 0|T[\hat{\Psi}_\alpha(x)\bar{\hat{\Psi}}_\beta(0)]0\rangle = i(\not{k} - m + i\epsilon)^{-1}_{\alpha\beta}$$
$$= \frac{i(\not{k} + m)_{\alpha\beta}}{k^2 - m^2 + i\epsilon} = \frac{i}{k^2 - m^2 + i\epsilon} \sum_s u(k,s)_\alpha \bar{u}(k,s)_\beta \qquad (57)$$

7. QFT III: — complex scalar, Dirac, Maxwell, EM interactions

Maxwell field

is a gauge field with $A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi$

- a plane wave solution to Maxwells equations $\partial_\mu F^{\mu\nu} = 0$
- giving the Maxwell Lagrangian $\mathcal{L}_{\text{em}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$$A^\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2E}} \sum_\lambda \left[\hat{\alpha}(k,\lambda) \varepsilon^\mu(k,\lambda) e^{-i\,k.x} + \hat{\alpha}^\dagger(k,\lambda) \varepsilon^{*\mu}(k,\lambda) e^{i\,k.x} \right] \quad (58)$$

- $\hat{\alpha}$ ($\hat{\alpha}^\dagger$) are the annihilation (creation) operators
- ε^μ are the polarisation vectors (polariation wave functions) fulfilling the conditions

$$k^2 = k.\varepsilon = 0 \;, \quad \varepsilon^2 = -1 \quad , \quad \boxed{\text{and}} \quad \varepsilon'^\mu = \varepsilon^\mu + \beta k^\mu \simeq \varepsilon^\mu \quad (59)$$

⇒ when $k^\mu = (k, 0, 0, k)$ the polarisations are only transverse:

$$\varepsilon^\mu(\lambda = +1) = -\frac{1}{\sqrt{2}}(0, 1, i, 0) \qquad \varepsilon^\mu(\lambda = -1) = \frac{1}{\sqrt{2}}(0, 1, -i, 0) \quad (60)$$

Maxwell field

- Quantizing A^μ is possible, but difficult
  - conditions not as operator equations, but as "weak" equations
    - only valid when evaluated on physical states

- the general propagator (in R_ξ gauge) is

$$\int d^4x\, e^{ik.x} \langle 0|T[A^\mu(x)A^\nu(0)]0\rangle = \frac{i}{k^2+i\epsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right] \tag{61}$$

- Electromagnetic interaction with the gauge principle

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu \tag{62}$$

giving
$$\widehat{H}_{\text{int}} = -\mathcal{L}_{\text{int}} = j^\mu_{em} A_\mu \tag{63}$$

with
$$j^\mu_{em} = q_\psi \bar{\Psi}\gamma^\mu\Psi + q_\phi i[\widehat{\phi}^\dagger(\partial^\mu\widehat{\phi}) - (\partial^\mu\widehat{\phi}^\dagger)\widehat{\phi}] - q_\phi^2 A^\mu \widehat{\phi}^\dagger\widehat{\phi} \tag{64}$$

or
$$\mathcal{L}_{\text{int}} = -q_\psi \bar{\Psi} \not{A} \Psi - q_\phi i[\widehat{\phi}^\dagger(\partial^\mu\widehat{\phi}) - (\partial^\mu\widehat{\phi}^\dagger)\widehat{\phi}]A_\mu + q_\phi^2 A_\mu A^\mu \widehat{\phi}^\dagger\widehat{\phi} \tag{65}$$

8. Elementary Processes in Scalar and Spinor Electrodynamics

Coulomb scattering: s^+ on static potential

- amplitude $\mathcal{A}_{s^+} = -i\int d^4x j^\mu_{em,s^+}(x)A_\mu(x)$
  - with $A^0 = \frac{Ze}{4\pi|\vec{x}|}$, $\vec{A} = 0$, and $j^\mu_{em,s^+}(x) = NN'(p+p')^\mu e^{-i(p-p').x}$

$$\mathcal{A}_{s^+} = -iNN'2\pi\delta(E-E')\frac{Ze^2}{|\vec{p}-\vec{p}\,'|^2}(E+E') = -i2\pi\delta(E-E')V_{s^+} \qquad (66)$$

- the transitionrate, eq.(28), becomes $\dot{P}_{s^+} = 2\pi\delta(E-E')|V_{s^+}|^2\rho(E')$
  - with the density of final states $\rho(E)dE = \frac{d^3\vec{p}}{(2\pi)^3 2E} = \frac{|\vec{p}\,|^2 d|\vec{p}\,|d\Omega}{16\pi^3 E} = \frac{|\vec{p}\,|dEd\Omega}{16\pi^3}$
    - there is no three-momentum conservation, as the static potential can absorb any recoil ...

- the cross section becomes $d\sigma = \frac{\dot{P}_{s^+}}{2|\vec{p}\,|} = \frac{2\pi|NN'Ze^2|\vec{p}-\vec{p}\,'|^{-2}2E|^2|\vec{p}\,'|d\Omega}{2|\vec{p}\,|16\pi^3}$
  - since $E' = E \;\Rightarrow\; |\vec{p}\,'| = |\vec{p}\,|$ and $|\vec{p}-\vec{p}\,'|^2 = 2|\vec{p}\,|^2(1-\cos\theta) = 4|\vec{p}\,|^2\sin^2\frac{\theta}{2}$
  - and taking $N = N' = 1$

$\Rightarrow$ Rutherford cross section

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2E^2}{4|\vec{p}\,|^2\sin^4\frac{\theta}{2}} \qquad (67)$$

8. Elementary Processes in Scalar and Spinor Electrodynamics

Coulomb scattering: e^- on static potential

- amplitude $\mathcal{A}_{e^-} = -i \int d^4x j^\mu_{em,e^-}(x) A_\mu(x)$
  - with $j^\mu_{em,e^-}(x) = -e\bar{\Psi}'(x)\gamma^\mu\Psi(x) \to -e\bar{u}(k',s')\gamma^\mu u(k,s)e^{-i(k-k').x}$

$$\mathcal{A}_{e^-} = -i2\pi\delta(E-E')\frac{Ze^2}{|\vec{q} = (\vec{k}-\vec{k}')|^2}u^\dagger(k',s')u(k,s) \tag{68}$$

- change from s^+ scattering:

$$2E \to u^\dagger(k',s')u(k,s) \tag{69}$$

  - averaging over initial spin and summing over final spin gives

$$\frac{1}{2}\sum_{s,s'}|u^\dagger(k',s')u(k,s)|^2 = (2E)^2(1 - \frac{|\vec{k}|^2}{E^2}\sin^2\frac{\theta}{2}) \tag{70}$$

Trace techniques

- when summing / averaging over spins
  - one can use the completeness relations eq.(52):

$$\sum_{s=\pm\frac{1}{2}} u_\alpha(k,s)\bar{u}_\beta(k,s) = (\not{k}+m)_{\alpha\beta} \qquad \sum_{s=\pm\frac{1}{2}} v_\alpha(k,s)\bar{v}_\beta(k,s) = (\not{k}-m)_{\alpha\beta} \quad (71)$$

- then the Lepton tensor

$$L^{\mu\nu}(k,k') := \frac{1}{2}\sum_{s,s'}[\bar{u}(k',s')\gamma^\mu u(k,s)][\bar{u}(k',s')\gamma^\nu u(k,s)]^* \quad (72)$$

becomes $2L^{\mu\nu}(k,k')$

$$\begin{aligned}
&= \sum_{s,s'}[\bar{u}(k',s')\gamma^\mu u(k,s)][\bar{u}(k',s')\gamma^\nu u(k,s)]^\dagger = \sum_{s,s'}\bar{u}(k',s')\gamma^\mu u(k,s)\bar{u}(k,s)\gamma^\nu u(k',s') \\
&= \sum_{s'}\bar{u}(k',s')\gamma^\mu(\not{k}+m)\gamma^\nu u(k',s') = \sum_{s'}[\gamma^\mu(\not{k}+m)\gamma^\nu]_{\alpha\beta}u(k',s')_\beta\bar{u}(k',s')_\alpha \\
&= \sum_{s'}[\gamma^\mu(\not{k}+m)\gamma^\nu]_{\alpha\beta}(\not{k}'+m)_{\beta\alpha} = \mathrm{Tr}[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}'+m)] \quad (73)
\end{aligned}$$

8. Elementary Processes in Scalar and Spinor Electrodynamics

Trace techniques

- using the Clifford algebra, eq.(13) $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} \times \mathbf{1}_{4\times 4}$ (74)
- and the cyclic property of the trace:

$$\mathrm{Tr}[AB\dots CD] = \mathrm{Tr}[B\dots CDA] = \mathrm{Tr}[CAB\dots D] \tag{75}$$

- one gets the result

$$\mathrm{Tr}[\mathbf{1}_{4\times 4}] = 4 \tag{76}$$

$$\mathrm{Tr}[\text{odd number of } \gamma\text{'s}] = 0 \tag{77}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = g^{\mu\nu}\mathrm{Tr}[\mathbf{1}_{4\times 4}] = 4g^{\mu\nu} \tag{78}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} + g^{\nu\rho}g^{\sigma\mu} - g^{\mu\rho}g^{\nu\sigma}) \tag{79}$$

- then the Lepton tensor becomes

$$\begin{aligned} L^{\mu\nu}(k,k') = \tfrac{1}{2}\mathrm{Tr}[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}'+m)] &= \tfrac{1}{2}m^2\mathrm{Tr}[\gamma^\mu\gamma^\nu] + \tfrac{1}{2}\mathrm{Tr}[\gamma^\mu\not{k}\gamma^\nu\not{k}'] \\ &= 2m^2g^{\mu\nu} + 2[k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}(k.k')] \qquad (80) \\ &= 2k^\mu k'^\nu + 2k^\nu k'^\mu + g^{\mu\nu}(k-k')^2 \qquad \text{using } k^2 = k'^2 = m^2 \end{aligned}$$

scattering $e^- s^+ \to e^- s^+$

- using the same current, eq.(64), in the interaction

$$\widehat{H}_{\text{int}} = \widehat{\jmath}^{\mu}_{em} \widehat{A}_{\mu} \tag{81}$$

- we have to calculate $\langle f | T e^{i \widehat{H}_{\text{int}}(x)} | i \rangle$
  - with

$$\langle f | = \langle e^-_{(k')} s^+_{(p')} | = (2E_{p'} 2E_{k'})^{1/2} \langle 0 | \widehat{c}_{s'}(k') \widehat{a}(p') \tag{82}$$

and

$$|i\rangle = | e^-_{(k)} s^+_{(p)} \rangle = (2E_p 2E_k)^{1/2} \widehat{c}^{\dagger}_s(k) \widehat{a}^{\dagger}(p) | 0 \rangle \tag{83}$$

- the first order in the expansion of $e^{i H_{\text{int}} t}$ has a single photon
  ⇒ it has to vanish:

$$\langle 0 | \widehat{\alpha} | 0 \rangle = \langle 0 | \widehat{\alpha}^{\dagger} | 0 \rangle = 0 \tag{84}$$

8. Elementary Processes in Scalar and Spinor Electrodynamics

scattering $e^- s^+ \to e^- s^+$

- the second order

$$\mathcal{A}_{e^-s^+} = \frac{(-i)^2}{2} \iint_{x,y} \langle 0 | \hat{c}_{s'}(k') \hat{a}(p') T[\hat{H}_{\text{int}}(x) \hat{H}_{\text{int}}(y)] \hat{c}_s^\dagger(k) \hat{a}^\dagger(p) | 0 \rangle \times (16 E_{p'} E_p E_{k'} E_k)^{1/2} \quad (85)$$

has the structure

$$\langle \hat{A}_\mu(x) \hat{A}_\mu(y) \rangle_T \langle s^+_{(p')}, e^-_{(k',s')} | (\hat{\jmath}^\mu_{em,s^+}(x) + \hat{\jmath}^\mu_{em,e^-}(x))(\hat{\jmath}^\nu_{em,s^+}(y) + \hat{\jmath}^\nu_{em,e^-}(y)) | s^+_{(p)}, e^-_{(k,s)} \rangle \quad (86)$$

– connecting the same current (s^+,e^-) to both points x,y disconnects the other:

$$\begin{aligned} \langle s^+_{(p')}, e^-_{(k',s')} | \hat{\jmath}^\mu_{em,s^+}(x) \hat{\jmath}^\nu_{em,s^+}(y) | s^+_{(p)}, e^-_{(k,s)} \rangle &= \langle s^+_{(p')} | \hat{\jmath}^\mu_{em,s^+}(x) \hat{\jmath}^\nu_{em,s^+}(y) | s^+_{(p)} \rangle \langle e^-_{(k',s')} | e^-_{(k,s)} \rangle \\ &= (2\pi)^3 \delta^3(\vec{k}' - \vec{k}) \delta_{s's} (2E_{p'} 2E_p)^{1/2} \\ &\quad \times \langle \hat{a}(p')(\hat{a}(q_1) + \hat{a}^\dagger(q_1))(\hat{a}(q_2) + \hat{a}^\dagger(q_2)) e^{i(q_1 - q_2).x} \\ &\quad\quad \times (\hat{a}(q_3) + \hat{a}^\dagger(q_3))(\hat{a}(q_4) + \hat{a}^\dagger(q_4)) e^{i(q_3 - q_4).x} \hat{a}^\dagger(p) \rangle_0 \end{aligned} \quad (87)$$

– which can be understood like p.160, Figure 6.3.(b) and (c)
  - an unconnected electron, ''flying through''
  - and a loop-correction on the s^+ line

- so the second order of $\mathcal{A}_{e^-s^+}$ reduces to

$$\frac{(-i)^2}{2} \iint_{x,y} \left\{ \langle s^+_{(p')} | \hat{\jmath}^\mu_{em}(x) | s^+_{(p)} \rangle \langle \hat{A}_\mu(x) \hat{A}_\mu(y) \rangle_T \langle e^-_{(k',s')} | \hat{\jmath}^\mu_{em}(x) | e^-_{(k,s)} \rangle + (x \leftrightarrow y) \right\} \quad (88)$$

8. Elementary Processes in Scalar and Spinor Electrodynamics

scattering $e^- s^+ \to e^- s^+$

- calculating the currents using the expansions eq.(39) and eq.(49):

$$\langle s^+_{(p')}|\hat{j}^\mu_{em}(x)|s^+_{(p)}\rangle = q_\phi(2E_{p'}2E_p)^{1/2}\langle\hat{a}(p')i[\hat{\phi}^\dagger(x)(\partial^\mu\hat{\phi}(x)) - (\partial^\mu\hat{\phi}^\dagger(x))\hat{\phi}(x)]\hat{a}^\dagger(p)\rangle_0$$

$$= iq_\phi(2E_{p'}2E_p)^{1/2}(2E_{q'}2E_q)^{-1/2}\iint_{q',q}\langle\hat{a}(p')\Big([\hat{a}^\dagger(q')e^{i\,q'.x} + \hat{b}(q')e^{-i\,q'.x}](-iq^\mu)[\hat{a}(q)e^{-i\,q.x} - \hat{b}^\dagger(q)e^{i\,q.x}]$$
$$-(iq'^\mu)[\hat{a}^\dagger(q')e^{i\,q'.x} - \hat{b}(q')e^{-i\,q'.x}][\hat{a}(q)e^{-i\,q.x} + \hat{b}^\dagger(q)e^{i\,q.x}]\Big)\hat{a}^\dagger(p)\rangle_0 \quad (89)$$

  – for a normal ordered Hamiltonian, $\hat{b}$ has to be on the right side of $\hat{b}^\dagger$:

$$\langle s^+_{(p')}|\hat{j}^\mu_{em}(x)|s^+_{(p)}\rangle = q_\phi(\frac{E_{p'}E_p}{E_{q'}E_q})^{-1/2}\iint_{q',q}\langle\hat{a}(p')\Big((q^\mu)[\hat{a}^\dagger(q')\hat{a}(q)e^{i\,(q'-q).x} - \hat{b}^\dagger(q)\hat{b}(q')e^{-i\,(q'-q).x}] \quad (90)$$
$$+(q'^\mu)[\hat{a}^\dagger(q')\hat{a}(q)e^{i\,(q'-q).x} - \hat{b}^\dagger(q)\hat{b}(q')e^{-i\,(q'-q).x}]\Big)\hat{a}^\dagger(p)\rangle_0$$
$$= q_\phi(\frac{E_{p'}E_p}{E_{q'}E_q})^{-1/2}\iint_{q',q}(q^\mu + q'^\mu)\langle\hat{a}(p')[\hat{a}^\dagger(q')\hat{a}(q)e^{i\,(q'-q).x} - \hat{b}^\dagger(q)\hat{b}(q')e^{-i\,(q'-q).x}]\hat{a}^\dagger(p)\rangle_0$$

  – using the commutators and the annihilation properties we get

$$\langle s^+_{(p')}|\hat{j}^\mu_{em}(x)|s^+_{(p)}\rangle = q_\phi\sqrt{\frac{E_{p'}E_p}{E_{q'}E_q}}\iint_{q',q}(q^\mu + q'^\mu)[(2\pi)^3\delta^3(\vec{p}\,' - \vec{q}\,')(2\pi)^3\delta^3(\vec{p} - \vec{q})e^{i\,(q'-q).x}]$$
$$= q_\phi(p^\mu + p'^\mu)e^{i\,(p'-p).x} \quad (91)$$

- similarly: $$\langle e^-_{(k',s')}|\hat{j}^\mu_{em}(x)|e^-_{(k,s)}\rangle = -q_\psi\bar{u}(k',s')\gamma^\mu u(k,s)e^{i\,(k'-k).x} \quad (92)$$

8. Elementary Processes in Scalar and Spinor Electrodynamics

scattering $e^- s^+ \to e^- s^+$

- together with the photon propagator eq.(93), Fourier transformed:

$$\langle A_\mu(x) A_\nu(y)\rangle_T = \int \frac{d^4q}{(2\pi)^4} \frac{i\, e^{-iq.(x-y)}}{q^2 + i\epsilon} \left[-g_{\mu\nu} + (1-\xi)\frac{q_\mu q_\nu}{q^2} \right] \tag{93}$$

- we get

$$\begin{aligned}
\mathcal{A}_{e^- s^+} &= \tfrac{(-i)^2}{2} \iint_{x,y} \int \frac{d^4q}{(2\pi)^4} \Big\{ q_\phi (p^\mu + p'^\mu) e^{i\,(p'-p).x} \tfrac{i}{q^2+i\epsilon} \left[-g_{\mu\nu} + (1-\xi)\tfrac{q_\mu q_\nu}{q^2} \right] e^{-iq.(x-y)} \\
&\qquad \times (-q_\psi) \bar{u}(k', s') \gamma^\nu u(k,s) e^{i\,(k'-k).y} + (x \leftrightarrow y) \Big\} \\
&= -\tfrac{q_\phi q_\psi}{2} \int \frac{d^4q}{(2\pi)^4} \iint_{x,y} \Big\{ e^{i\,(p'-p-q).x} e^{i\,(k'-k+q).y} \\
&\qquad \times \tfrac{i}{q^2+i\epsilon} \left[-(p_\nu + p'_\nu) + (1-\xi)\tfrac{((p+p').q) q_\nu}{q^2} \right] \bar{u}(k',s') \gamma^\nu u(k,s) + (x \leftrightarrow y) \Big\} \\
&= -q_\phi q_\psi \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta^4(p' - p - q) (2\pi)^4 \delta^4(k' - k + q) \\
&\qquad \times \tfrac{i}{q^2+i\epsilon} \bar{u}(k', s') \left[-(\not{p} + \not{p}') + (1-\xi) \tfrac{((p+p').q)}{q^2} \not{q} \right] u(k,s)
\end{aligned} \tag{94}$$

– using the δ-function we see: $q = p' - p \quad \Rightarrow \quad (p+p').q = p'^2 - p^2 = m_s^2 - m_s^2 = 0$

⇒ no gauge dependence (dependence on ξ) in $\mathcal{A}_{e^- s^+}$

scattering $e^- s^+ \to e^- s^+$

with the definition of the scattering matrix, eq.(26),

$$\mathcal{A}_{e^-s^+}(s,s') = i(2\pi)^4\delta^4(p'+k'-p-k)\mathcal{M}_{e^-s^+}(s,s') \tag{95}$$

- we get the invariant amplitude (with $q = k - k'$)

$$\mathcal{M}_{e^-s^+}(s,s') = -q_\psi \bar{u}(k',s')\gamma^\mu u(k,s)\,\frac{-g_{\mu\nu}}{q^2+i\epsilon}\,q_\phi(p+p')^\nu \tag{96}$$

- and the differential cross section

$$d\sigma_{e^-s^+}(s,s') = \frac{|\mathcal{M}_{e^-s^+}(s,s')|^2}{4E_pE_k|\vec{v}_{\text{rel}}|}d\text{Lips}(s=(p+k)^2;p',k') \tag{97}$$

- for the unpolarised cross section we have to sum / average over spins:

$$\langle d\sigma_{e^-s^+}\rangle = \frac{\frac{1}{2}\sum_{s,s'}|\mathcal{M}_{e^-s^+}(s,s')|^2}{4[(p.k)^2-(m_{s^+}m_{e^-})^2]^{1/2}}d\text{Lips}(s;p',k') \tag{98}$$

scattering $e^- s^+ \to e^- s^+$

- summing / averaging over electron spins gave the Lepton tensor, eq.(72)

$$L^{\mu\nu}(k,k') := \frac{1}{2}\sum_{s,s'}[\bar{u}(k',s')\gamma^\mu u(k,s)][\bar{u}(k',s')\gamma^\nu u(k,s)]^* = 2k^\mu k'^\nu + 2k^\nu k'^\mu + g^{\mu\nu}(k-k')^2 \tag{99}$$

- we get with $k - k' = q = p' - p$

$$\frac{1}{2}\sum_{s,s'}|\mathcal{M}_{e^-s^+}(s,s')|^2 = (\tfrac{q_\psi q_\phi}{q^2+i\epsilon})^2[2k^\mu k'^\nu + 2k^\nu k'^\mu + g^{\mu\nu}q^2](p+p')_\mu(p+p')_\nu \tag{100}$$

  - but the Lepton tensor was transverse: $L^{\mu\nu}q_\mu = L^{\mu\nu}q_\nu = 0$

$$[2k^\mu k'^\nu + 2k^\nu k'^\mu + g^{\mu\nu}q^2](k-k')_\nu = 2k^\mu(k.k' - k'^2) + 2(k^2 - k.k')(k^\mu - q^\mu) + q^2 q^\mu$$
$$= 2k^\mu(k.k' - m^2 + m^2 - k.k')k^\mu - q^\mu(2m^2 - 2k.k' - (k-k')^2) = 0 \tag{101}$$

- so $$\frac{1}{2}\sum_{s,s'}|\mathcal{M}_{e^-s^+}(s,s')|^2 = (\tfrac{q_\psi q_\phi}{q^2+i\epsilon})^2[2k^\mu k'^\nu + 2k^\nu k'^\mu + g^{\mu\nu}q^2](2p+q)_\mu(2p+q)_\nu$$
$$= (\tfrac{q_\psi q_\phi}{q^2+i\epsilon})^2 4[4k.p\,k'.p + p^2q^2] = (\tfrac{q_\psi q_\phi}{q^2+i\epsilon})^2 8[2k.p\,k'.p + \tfrac{q^2}{2}M_{s^+}^2] \tag{102}$$

scattering $e^- s^+ \to e^- s^+$

- for the cross section in the CM frame we still need
  - the phasespace in the CM frame, eq.(32)

$$d\text{Lips}(s; p_A, p_B)_{CM} = \frac{d\Omega}{4\pi} \frac{p}{4\pi W} dW \delta(\sqrt{s} - W) \tag{103}$$

  - and the fluxfactor in the CM frame, eq.(35)

$$4\sqrt{(p_A.p_B)^2 - (m_A m_B)^2} = 4p\sqrt{s} \tag{104}$$

- finally we get with $q_\phi = -q_\psi = e$ and

$$\begin{aligned} \left(\frac{\langle d\sigma_{e^- s^+}\rangle}{d\Omega}\right)_{\text{CM}} &= (\tfrac{-e^2}{q^2})^2 \frac{8[2k.p\,k'.p + \frac{q^2}{2} M_{s^+}^2]}{4p\sqrt{s}} \frac{1}{4\pi} \frac{p}{4\pi\sqrt{s}} \\ &= \frac{2\alpha^2}{s(q^2)^2} [2k.p\,k'.p + \tfrac{q^2}{2} M_{s^+}^2] \end{aligned} \tag{105}$$

  - all quantities are invariant — except the solid angle $d\Omega$

8. Elementary Processes in Scalar and Spinor Electrodynamics

scattering $e^- s^+ \to e^- s^+$

- we can boost the CM-frame into the Lab-frame
  - the invariant phasespace element ist still $\frac{d^3\vec{k}'}{(2\pi)^3 2E'_k}$
  - but the solid angle $d\Omega$ changes
- with the limit $m_e \ll \sqrt{s}$ we ignore the electron mass: $k^2 = k'^2 = 0$
  - we get $q^2 = (k - k')^2 = -2k.k' = -2E_k E'_k(1 - \cos\theta) = -4E_k E'_k \sin^2\frac{\theta}{2}$
  - with $p^\mu_{\text{Lab}} = (M_{s^+}, \vec{0})$ we get $k.p = M_{s^+}E_k$ and $k'.p = M_{s^+}E'_k$
  - and $s = (p+k)^2 = p^2 + 2p.k + k^2 = M^2_{s^+} + 2M_{s^+}E_k = M_{s^+}(M_{s^+} + 2E_k)$
- elastic scattering implies $p'^2 = M^2_{s^+}$ or

$$\begin{aligned} 0 &= M^2_{s^+} - (p + k - k')^2 = -2p.k + 2p.k' + 2k.k' = 4E_k E'_k \sin^2\frac{\theta}{2} + 2M_{s^+}E'_k - 2M_{s^+}E_k \\ &= 2M_{s^+}E'_k\left(2\frac{E_k}{M_{s^+}}\sin^2\frac{\theta}{2} + 1 - \frac{E_k}{E'_k}\right) \quad \Rightarrow \quad \frac{E_k}{E'_k} = 1 + 2\frac{E_k}{M_{s^+}}\sin^2\frac{\theta}{2} \end{aligned} \tag{106}$$

- giving the no-structure cross section

$$\left(\frac{\langle d\sigma_{e^-s^+}\rangle}{d\Omega}\right)_{\text{ns}} = \frac{\alpha^2}{4E_k^2\sin^4\frac{\theta}{2}} \cdot \frac{E'_k}{E_k}\cos^2\frac{\theta}{2} \tag{107}$$

Formfactors

- a static charge distribution has a potential $A^0(x)$,
  - following the Poisson equation $\vec{\nabla}^2 A^0(x) = -Ze\rho(x)$

- calculating the scattering on the Coulomb potential
  - we were using the Fourier transform $\tilde{A}^0(\vec{q}) = \int_x e^{-i\vec{q}\cdot\vec{x}} A^0(x)$
  - taking the Fourier transformation of the Poisson equation we get

$$\int_x e^{-i\vec{q}\cdot\vec{x}} \vec{\nabla}^2 A^0(x) = -Ze \int_x e^{-i\vec{q}\cdot\vec{x}} \rho(x) =: -ZeF(\vec{q}) \tag{108}$$

  - doing two times a partial integration we get

$$\int_x e^{-i\vec{q}\cdot\vec{x}}\, \vec{\nabla}^2 A^0(x) = \int_x \vec{\nabla}^2 e^{-i\vec{q}\cdot\vec{x}} A^0(x) = \int_x -(\vec{q})^2 e^{-i\vec{q}\cdot\vec{x}} A^0(x) = -\vec{q}^{\,2} \tilde{A}^0(\vec{q}) \tag{109}$$

- taking an exponential shape for the density $\rho(x) = \frac{e^{-|\vec{x}|/a}}{[8\pi a^3]^{1/2}}$
  - we get $F(\vec{q}) = \frac{1}{[(\vec{q})^2 a^2 + 1]^2}$

π^+ formfactor

- we want to generalize the conserved current of the s^+

$$j^{\mu}_{em,s^+}(x) = \langle s^+_{(p')}|\hat{j}^{\mu}_{em}(x)|s^+_{(p)}\rangle = q_\phi (p+p')^\mu e^{i\,(p'-p).x} \tag{110}$$

- two independent fourvectors: p^μ and p'^μ (or $(p+p')^\mu$ and $q^\mu = (p'-p)^\mu$)
  - but only one independent scalar $q^2 = 2M^2 - 2p.p'$, since $p^2 = p'^2 = M^2$
- we can make an ansatz with two formfactors, depending on q^2

$$j^{\mu}_{em,\pi^+}(x) = \langle \pi^+_{(p')}|\hat{j}^{\mu}_{em}(x)|\pi^+_{(p)}\rangle =: e[F(q^2)(p+p')^\mu + G(q^2)q^\mu]e^{i\,q.x} \tag{111}$$

- current conservation $\partial_\mu j^\mu = 0$ gives us

$$q_\mu[F(q^2)(p+p')^\mu + G(q^2)q^\mu] = [F(q^2)q.(p+p') + G(q^2)q^2] = G(q^2)q^2 = 0 \tag{112}$$

  - giving us the constraint: $G(q^2) \propto \delta(q^2)$

8. Elementary Processes in Scalar and Spinor Electrodynamics

scattering $e^-\pi^+ \to e^-\pi^+$

- one can study the pion formfactor with the scattering

$$e^-(k,s) + \pi^+(p) \to e^-(k',s') + \pi^+(p') \tag{113}$$

  – this probes the formfactor in the space-like region: $q^2 = (k-k')^2 < 0$

- studying the process

$$e^+(k_1,s_1) + e^-(k,s) \to \pi^+(p') + \pi^-(p_1) \tag{114}$$

  gives the same matrix element as

$$e^-(k,s) + \pi^+(-p_1) \to e^-(-k_1,-s_1) + \pi^+(p') \tag{115}$$

  – this probes the formfactor in the time-like region:

$$q^2 = (k-k')^2 = (k-(-k_1))^2 = (p'+p_1)^2 > (2m_\pi)^2 \tag{116}$$

$\Rightarrow$ crossing symmetry $\equiv$ **CPT**-symmetry

8. Elementary Processes in Scalar and Spinor Electrodynamics

Electron Compton scattering

$$\gamma(k,\lambda) + e^-(p,s) \to \gamma(k',\lambda') + e^-(p',s') \tag{117}$$

- the factors for external photons are obtained "as usual"
  - using the plane wave expansion, eq. (58)
  - sandwiching field with creation/anihilation operator in the vacuum:

$$\sqrt{2E'_k}\langle 0|\hat{\alpha}(k,\lambda)A^\mu(x)|0\rangle = \varepsilon^{*\mu}(k,\lambda)e^{i\,k.x} \tag{118}$$

- the fermion propagator was given in eq. (57)
- the amplitude for the two diagrams (Figure 8.14) becomes

$$\begin{aligned}\mathcal{M}_{\gamma e^-}(s,s',\lambda,\lambda') &= -e^2\varepsilon^{*\mu}(k',\lambda')\varepsilon^\nu(k,\lambda) \\ &\quad\times\bar{u}(p',s')\left[\gamma_\mu\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\gamma_\nu + \gamma_\nu\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\gamma_\mu\right]u(p,s) \\ &= -e^2\varepsilon^{*\mu}(k',\lambda')\varepsilon^\nu(k,\lambda)\bar{u}(p',s')\left[\gamma_\mu\frac{\not{p}+\not{k}+m}{2p.k}\gamma_\nu + \gamma_\nu\frac{\not{p}-\not{k}'+m}{-2p.k'}\gamma_\mu\right]u(p,s)\end{aligned} \tag{119}$$

Electron Compton scattering

- gauge invariance requires the invariance

$$\varepsilon^\mu(k,\lambda) \sim \varepsilon'^\mu(k,\lambda) = \varepsilon^\mu(k,\lambda) + \beta k^\mu \tag{120}$$

$\Rightarrow$ replacing any $\varepsilon^\mu(k,\lambda)$ with k^μ in $\mathcal{M}_{\gamma e^-}$ has to give 0 (Ward identity)

- checking:

(with $k-k'=p'-p$, $k^2=k'^2=0$, $p^2=p'^2=m^2$ and $p.k'=p'.k$, $p'.k'=p.k$)

$$\begin{aligned}
& k'^\mu \bar{u}(p',s')\left[\gamma_\mu \frac{\not p+\not k+m}{2p.k}\gamma_\nu + \gamma_\nu \frac{\not p-\not k'+m}{-2p.k'}\gamma_\mu\right] u(p,s) \\
= \;& \bar{u}(p',s')\left[\frac{\not k'(\not p'+\not k'+m)\gamma_\nu}{2p.k} - \frac{\gamma_\nu(\not p-\not k'+m)\not k'}{2p.k'}\right] u(p,s) \\
= \;& \bar{u}(p',s')\left[\frac{(\{\not k',\not p'\}+[-\not p'+0+m]\not k')\gamma_\nu}{2p.k} - \frac{\gamma_\nu(\{\not k',\not p\}+\not k'[-\not p-0+m])}{2p.k'}\right] u(p,s) \\
= \;& \bar{u}(p',s')\left[\frac{2k'.p'\gamma_\nu}{2p.k} - \frac{\gamma_\nu 2k'.p}{2p.k'}\right] u(p,s) \;=\; \left(\frac{k'.p'}{p.k} - \frac{k'.p}{p.k'}\right)\bar{u}(p',s')\gamma_\nu u(p,s) \\
= \;& (1-1)\bar{u}(p',s')\gamma_\nu u(p,s) \;=\; 0
\end{aligned} \tag{121}$$

Electron Compton scattering

- gauge invariance simplifies the spinsum for external photons
  - any matrixelement with an external photon line can be written as

$$\mathcal{M} = \varepsilon^{\mu}(k,\lambda) T_{\mu} \tag{122}$$

  - for the unpolarized cross section we get

$$\sum_{\lambda=\pm1} |\mathcal{M}|^2 = \sum_{\lambda=\pm1} \varepsilon^{\mu}(k,\lambda) T_{\mu} \varepsilon^{\nu *}(k,\lambda) T_{\nu}^{*} = |T_1|^2 + |T_2|^2 \tag{123}$$

  - from the Ward identity we had (with $k^{\mu} = (k,0,0,k)$ and $T^{\mu} := (T_0, T_1, T_2, T_3)$)

$$0 = k^{\mu} T_{\mu} = k T_0 - k T_3 \quad \Rightarrow \quad T_0 = T_3 \tag{124}$$

    and hence

$$\sum_{\lambda=\pm1} \varepsilon^{\mu}(k,\lambda) T_{\mu} \varepsilon^{\nu *}(k,\lambda) T_{\nu}^{*} = |T_1|^2 + |T_2|^2 + |T_3|^2 - |T_0|^2 = -g^{\mu\nu} T_{\mu} T_{\nu}^{*} \tag{125}$$

$\Rightarrow$ photon spin sum $\displaystyle\sum_{\lambda=\pm1} \varepsilon^{\mu}(k,\lambda) \varepsilon^{\nu *}(k,\lambda) = -g^{\mu\nu}$ (126)

Electron Compton scattering

- we get the unpolarized cross section $\langle|\mathcal{M}_{\gamma e^-}|^2\rangle$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{\lambda,\lambda',s,s'} e^4 \varepsilon'^{*\mu}\varepsilon^{\nu}\bar{u}'\left[\gamma_\mu\frac{\not{p}+\not{k}+m}{2p.k}\gamma_\nu - \gamma_\nu\frac{\not{p}-\not{k}'+m}{2p.k'}\gamma_\mu\right] u\varepsilon'^{\mu'}\varepsilon^{*\nu'}\bar{u}\left[\gamma_{\nu'}\frac{\not{p}+\not{k}+m}{2p.k}\gamma_{\mu'} - \gamma_{\mu'}\frac{\not{p}-\not{k}'+m}{2p.k'}\gamma_{\nu'}\right] u' \\
&= \frac{e^4(-g^{\mu\mu'})(-g^{\nu\nu'})}{16}\mathrm{Tr}\left[\left(\gamma_\mu\frac{\not{q}+m}{p.k}\gamma_\nu - \gamma_\nu\frac{\not{r}+m}{p.k'}\gamma_\mu\right)(\not{p}-m)\left(\gamma_{\nu'}\frac{\not{q}+m}{p.k}\gamma_{\mu'} - \gamma_{\mu'}\frac{\not{r}+m}{p.k'}\gamma_{\nu'}\right)(\not{p}'-m)\right] \\
&= \frac{e^4}{16}\mathrm{Tr}\left[\left(\gamma_\mu\frac{\not{q}+m}{p.k}\gamma_\nu - \gamma_\nu\frac{\not{r}+m}{p.k'}\gamma_\mu\right)(\not{p}-m)\left(\gamma^{\nu}\frac{\not{q}+m}{p.k}\gamma^{\mu} - \gamma^{\mu}\frac{\not{r}+m}{p.k'}\gamma^{\nu}\right)(\not{p}'-m)\right] \\
&= \frac{e^4}{16}\Bigg(\mathrm{Tr}\left[\gamma_\mu\frac{\not{q}+m}{p.k}\gamma_\nu(\not{p}-m)\gamma^\nu\frac{\not{q}+m}{p.k}\gamma^\mu(\not{p}'-m)\right] + \mathrm{Tr}\left[\gamma_\nu\frac{\not{r}+m}{p.k'}\gamma_\mu(\not{p}-m)\gamma^\mu\frac{\not{r}+m}{p.k'}\gamma^\nu(\not{p}'-m)\right] \\
&\qquad -\mathrm{Tr}\left[\gamma_\mu\frac{\not{q}+m}{p.k}\gamma_\nu(\not{p}-m)\gamma^\mu\frac{\not{r}+m}{p.k'}\gamma^\nu(\not{p}'-m) + \gamma_\nu\frac{\not{r}+m}{p.k'}\gamma_\mu(\not{p}-m)\gamma^\nu\frac{\not{q}+m}{p.k}\gamma^\mu(\not{p}'-m)\right] \\
&= \frac{e^4}{16}(A+B-C)
\end{aligned}
\tag{127}
$$

  - using the abbreviations: $u^{(\prime)} = u(p^{(\prime)}, s^{(\prime)})$, $\varepsilon^{(\prime)[*]\mu} = \varepsilon^{[*]\mu}(k^{(\prime)}, \lambda^{(\prime)})$
  - and definitions $q^\mu = (p+k)^\mu$ and $r^\mu = (p-k')^\mu$
    - connected to the Mandelstam variables

$$s = q^2 = (p+k)^2 = p^2 + 2p.k + k^2 = m^2 + 2p.k \quad\Rightarrow\quad p.k = \tfrac{1}{2}(s-m^2) \tag{128}$$

$$u = r^2 = (p-k')^2 = p^2 - 2p.k' + k'^2 = m^2 - 2p.k' \quad\Rightarrow\quad p.k' = -\tfrac{1}{2}(u-m^2) \tag{129}$$

8. Elementary Processes in Scalar and Spinor Electrodynamics

Electron Compton scattering

- using $\gamma_\mu \not{a} \gamma^\mu = 2a_\mu \gamma^\mu - \not{a}\gamma_\mu\gamma^\mu = \not{a}(2-D) = -2\not{a}$ we get for A

$$A = \mathrm{Tr}\left[\gamma_\mu \frac{\not{q}+m}{p.k}\gamma_\nu(\not{p}-m)\gamma^\nu \frac{\not{q}+m}{p.k}\gamma^\mu(\not{p}'-m)\right]$$
$$= \mathrm{Tr}\left[\frac{\not{q}+m}{p.k}(-2\not{p}-4m)\frac{\not{q}+m}{p.k}(-2\not{p}'-4m)\right] \qquad (130)$$

  - motivated by later discussed quark-gluon scattering, we set $m \to 0$

$$A = \frac{4}{(p.k)^2}\mathrm{Tr}\left[\not{q}\not{p}\not{q}\not{p}'\right] = \frac{4}{(p.k)^2}\mathrm{Tr}\left[(\not{p}+\not{k})\not{p}(\not{p}+\not{k})\not{p}'\right] = \frac{4}{(p.k)^2}\mathrm{Tr}\left[\not{k}\not{p}\not{k}\not{p}'\right]$$
$$= \frac{16}{(p.k)^2}\left[2p.k\,p'.k - k^2\,p.p'\right] = 32\frac{-u}{s} \qquad (131)$$

- B is like A, just with the replacements: $\not{q} \to \not{r}$ and $p.k \to p.k'$
  - or $s \leftrightarrow (-u)$

$$B = 32\frac{s}{-u} \qquad (132)$$

Electron Compton scattering

- for C we get (using $\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$ and $\gamma_\mu \not{a}\not{b}\gamma^\mu = 4a.b$)

$$\begin{aligned} C &= \text{Tr}\left[\gamma_\mu \frac{\not{q}}{p.k}\gamma_\nu \not{p}\gamma^\mu \frac{\not{r}}{p.k'}\gamma^\nu \not{p}' + \gamma_\nu \frac{\not{r}}{p.k'}\gamma_\mu \not{p}\gamma^\nu \frac{\not{q}}{p.k}\gamma^\mu \not{p}'\right] && (133) \\ &= \frac{1}{p.k}\frac{1}{p.k'}\text{Tr}\left[\not{p}\gamma_\nu \not{q}\not{r}\gamma^\nu \not{p}' + \gamma_\nu \not{r}\not{q}\gamma^\nu \not{p}\not{p}'\right] && (134) \\ &= \frac{4(q.r)}{(p.k)\,(p.k')}\text{Tr}\left[\not{p}\not{p}' + \not{p}\not{p}'\right] = \frac{32(q.r)(p.p')}{(p.k)\,(p.k')} \end{aligned}$$

  - but

$$\begin{aligned} q.r &= (p+k).(p-k') = p^2 + p.k - (p+k).k' && (135) \\ &= 0 + p.k - (p'+k').k' = p.k - p'.k' - k'^2 = p.k - p.k - 0 = 0 \end{aligned}$$

$$\Rightarrow \quad \langle|\mathcal{M}_{\gamma e^-}|^2\rangle = \frac{e^4}{16}(32\frac{-u}{s} + 32\frac{s}{-u} - 0) = -2e^4(\frac{u}{s} + \frac{s}{u}) \qquad (136)$$

and with eq.(36)

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{CM}} = \frac{|\mathcal{M}_{fi}|^2}{(8\pi)^2 s} = \frac{\alpha^2}{2s}(\frac{-u}{s} + \frac{s}{-u}) \qquad (137)$$

- for loosening a constraint (like $k^2 = -Q^2$)
  - one has to start the calculation from the beginning!
    - ∗ before the constraint was used . . .

Electron muon elastic scattering

- assuming one photon exchange

$$\mathcal{M}_{e^-\mu^-}(r,s;r',s') = e\bar{u}(k',s')\gamma_\mu u(k,s)\frac{-g^{\mu\nu}}{q^2}e\bar{u}(p',r')\gamma_\nu u(p,r) \tag{138}$$

and
$$\langle|\mathcal{M}_{e^-\mu^-}|^2\rangle = \left(\frac{e^2}{q^2}\right)^2 L_{\mu\rho}(k,k')g^{\mu\nu}g^{\rho\sigma}M_{\nu\sigma}(p,p') \tag{139}$$

with $M_{\nu\sigma}$ the leptonic current of the muon.

- remembering $L_{\mu\nu}$ is symmetric, $2p.k = (p+k)^2 - p^2 - k^2$, $2p.k' = p^2 + k'^2 - (p-k')^2$, etc

$$\begin{aligned} L_{\mu\nu}M^{\mu\nu} &= 2[k_\mu k'_\nu + k_\nu k'_\mu + \tfrac{q^2}{2}g_{\mu\nu}]2[2p^\mu p'^\nu + \tfrac{q^2}{2}g^{\mu\nu}] \\ &= 4[2p.k\,p'.k' + 2p.k'\,p.k' + q^2 p.p' + q^2 k.k' + (\tfrac{q^2}{2})^2 4] \\ &= 2[(2p.k)^2 + (2p.k')^2 + q^2[2p^2 - (p-p')^2] + q^2[2k^2 - (k-k')^2] + 2(q^2)^2] \\ &= 2[(2p.k)^2 + (2p.k')^2 + 2q^2(M^2 + m^2)] \end{aligned} \tag{140}$$

  - compare to eq.(105)

- going to the muon rest frame, i.e. $p^\mu = (M,0,0,0)$, we get ... (no proof)

$$\frac{\langle d\sigma_{e^-\mu^-}\rangle}{d\Omega} = \left(\frac{\langle d\sigma_{e^-s^+}\rangle}{d\Omega}\right)_{\text{ns}}\left(1 - \frac{q^2}{2M^2}\tan^2\frac{\theta}{2}\right) \tag{141}$$

Electron proton elastic scattering and nucleon form factors

- writing a conserved proton current in analogy to the electron current

$$J^\mu_{\mathsf{p}}(p,p';s,s') = \langle \mathsf{p};p',s'|\hat{j}^\mu_{em,p}|\mathsf{p};p,s\rangle \quad \text{with} \quad q_\mu J^\mu_{\mathsf{p}}(p,p') = 0 \tag{142}$$

  we can define a proton tensor

$$B^{\mu\nu} = \frac{1}{2}\sum_{s,s'}\langle \mathsf{p};p',s'|\hat{j}^\mu_{em,p}|\mathsf{p};p,s\rangle\langle \mathsf{p};p',s'|\hat{j}^\nu_{em,p}|\mathsf{p};p,s\rangle^* \tag{143}$$

- Lorentz invariance of the cross section requires that $B^{\mu\nu}$ is a tensor
  - can be constructed from p^μ, $q^\mu = (p'-p)^\mu$, and $g^{\mu\nu}$
    * Parity forbids the term $\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta$
- current conservation requires $q_\mu B^{\mu\nu} = q_\nu B^{\mu\nu} = 0$

  ⇒ find tensors from p^μ, $q^\mu = (p'-p)^\mu$, and $g^{\mu\nu}$
    * easiest to construct from vectors, that are normal to q_μ
  - solution: define $\tilde{p}^\mu = p^\mu - \frac{p.q}{q^2}q^\mu$, then

$$B^{\mu\nu} = 4A(q^2)\tilde{p}^\mu\tilde{p}^\nu + 2M^2B(q^2)[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}] \tag{144}$$

8. Elementary Processes in Scalar and Spinor Electrodynamics

Electron proton elastic scattering and nucleon form factors

- the traditional definition of the proton current is

$$\langle \mathrm{p}; p', s' | \hat{j}^{\mu}_{em,p} | \mathrm{p}; p, s \rangle = (+e)\bar{u}(p', s') \left[\gamma^{\mu} \mathcal{F}_1(q^2) + \frac{i\kappa \mathcal{F}_2(q^2)}{2M} \sigma^{\mu\nu} q_{\nu} \right] u(p, s) \quad (145)$$

  - with the normalisations $\mathcal{F}_1(0) = \mathcal{F}_1(0) = 1$
  - the spin tensor $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$
  - and the anomalous magnetic moment $\mu_p = 1 + \kappa$ with $\kappa = 1.79$

- the cross section in the Laboratory frame, $p^{\mu} = (M, 0, 0, 0)$ is

$$\frac{\langle d\sigma_{e^- p^+} \rangle}{d\Omega} = \left(\frac{\langle d\sigma_{e^- s^+} \rangle}{d\Omega} \right)_{\text{ns}} \left(A + B \tan^2 \frac{\theta}{2} \right) \quad (146)$$

with

$$A = \mathcal{F}_1^2 + \tau\kappa^2 \mathcal{F}_2^2 \quad (147)$$

$$B = 2\tau(\mathcal{F}_1 + \kappa \mathcal{F}_2)^2 \quad (148)$$

and the kinematic factor

$$\tau = \frac{-q^2}{4M^2} \quad (149)$$

Kinematic of electron proton scattering

- looking at the point like case we take μ^+ instead of p
- energy-momentum conservation requires $$p_\mu + q_\mu = p'_\mu \tag{150}$$
- mass-shell condition gives $$p^2 = p'^2 = M^2 \tag{151}$$

⇒ elastic scattering $$2p.q = -q^2 \tag{152}$$

- Laboratory frame, $p^\mu = (M, 0, 0, 0)$: same definitions as on slide p.31
  - additionally: $$p.q = M(E_k - E'_k) =: M\nu \quad (\text{ so } \nu = q^0) \tag{153}$$
  - with $Q^2 = -q^2$ elastic scattering becomes $$\nu = \frac{Q^2}{2M} \tag{154}$$
  - changing variables $$d\Omega = 2\pi d(\cos\theta) = (\pi/E'^2_k)dQ^2 \tag{155}$$
  - gives the double-differential cross section in the Lab frame

$$\frac{d^2\sigma}{dQ^2\, d\nu} = \frac{\pi\alpha^2}{4E_k^2 \sin^4\frac{\theta}{2}} \cdot \frac{1}{E_k E'_k}\left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2}\sin^2\frac{\theta}{2}\right]\delta\left(\nu - \frac{Q^2}{2M}\right) \tag{156}$$

Inelastic electron-proton scattering: structure functions

- generalising the proton tensor, eq.(143), to inelastic scattering
  - we have to sum over all possible final states

$$e^2 W^{\mu\nu}(q,p) = \frac{(2\pi)^4\delta^4(p+q-p')}{8\pi M}\frac{1}{2}\sum_{s,X}\langle \mathrm{p};p,s|\hat{j}^{\mu}_{em,p}|X;p'\rangle\langle X;p'|\hat{j}^{\nu}_{em,p}|\mathrm{p};p,s\rangle \quad (157)$$

  - but the invariant mass of the final state is variable: $p'^2 = W^2$
  - parametrizing $W^{\mu\nu}$ by the same tensors
    - ∗ with the definition $\tilde{p}^{\mu} = p^{\mu} - \frac{p.q}{q^2}q^{\mu}$
  - and functions depending on Q^2 and $\nu = q^0|_{\mathrm{CM}}$

$$W^{\mu\nu}(q,p) = (-g^{\mu\nu} + \frac{q^{\mu}q^{\nu}}{q^2})W_1(Q^2,\nu) + \frac{\tilde{p}^{\mu}\tilde{p}^{\nu}}{M^2}W_2(Q^2,\nu) \quad (158)$$

- the total double-differential cross section becomes – compare eq.(156)

$$\frac{d^2\sigma}{dQ^2\,d\nu} = \frac{\pi\alpha^2}{4E_k^2\sin^4\frac{\theta}{2}}\cdot\frac{1}{E_kE'_k}\left[W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2}\right] \quad (159)$$

9. Deep Inelastic Electro-Nucleon Scattering and the Quark Parton Model

Bjorken scaling and the parton model

- in the deep inelastic region

$$\left.\begin{array}{r} Q^2 \to \infty \\ \nu \to \infty \end{array}\right\} \text{ with } x = \frac{Q^2}{2M\nu} \text{ fixed } \Rightarrow \begin{array}{r} MW_1(Q^2,\nu) \to F_1(x) \\ \nu W_2(Q^2,\nu) \to F_2(x) \end{array} \tag{160}$$

$\Rightarrow$ F_1 and F_2 finite

- the scaling can be understood,
  - if the scattering happens on point-like ''partons'' inside the proton
  - each parton carries a fraction f of the proton momentum P^μ

$$p_i^\mu = f P^\mu \tag{161}$$

  - on-shell condition for a single parton means

$$p_i^2 (= f^2 P^2) = p_i'^2 = (p_i + q)^2 = (f P^\mu + q)^2 = f^2 P^2 + 2 f P.q + q^2 \tag{162}$$

    * with $Q^2 = -q^2$ and $P.q = Mq^0 = M\nu$

$\Rightarrow$ the fraction f is the Bjorken-x: $\quad f = \dfrac{Q^2}{2M\nu} = x_B \qquad (163)$

9. Deep Inelastic Electro-Nucleon Scattering and the Quark Parton Model

Bjorken scaling and the parton model

- understanding the proton as a superposition of incoherent partons
  - compare the double-differential cross sections (156) and (159)
    * with $\nu = \frac{Q^2}{2Mx}$ and e_i the electric charge of the parton i

$$W_1^i(Q^2, x) = e_i^2 \frac{Q^2}{4M^2x^2}\delta(\nu - \frac{Q^2}{2Mx}) \quad (164)$$

$$W_2^i(Q^2, x) = e_i^2 \delta(\nu - \frac{Q^2}{2Mx}) \quad (165)$$

- for full scattering of the proton we have to sum over all partons
  - with a probability $f_i(x)$ to find parton i with momentum fraction x

$$W_n = \sum_i \int_0^1 dx\, f_i(x) W_n^i \quad (166)$$

or

$$W_1(Q^2, \nu) = \sum_i \int_0^1 dx\, f_i(x) e_i^2 \frac{Q^2}{4M^2x^2}\delta(\nu - \frac{Q^2}{2Mx}) = \frac{1}{M}\sum_i \frac{1}{2} e_i^2 f_i(x_B) = \frac{1}{M} F_1(x_B) \quad (167)$$

$$W_2^i(Q^2, \nu) = \sum_i \int_0^1 dx\, f_i(x) e_i^2 \delta(\nu - \frac{Q^2}{2Mx}) = \frac{1}{\nu}\sum_i e_i^2 x_B f_i(x_B) = \frac{1}{\nu} F_2(x_B) \quad (168)$$

Bjorken scaling and the parton model

- from this representation for F_1 and F_2 follows

$$F_2(x) = \sum_i e_i^2 x f_i(x) = 2x \sum_i \tfrac{1}{2} e_i^2 f_i(x) = 2x F_1(x) \tag{169}$$

$\Rightarrow$ Callan-Gross relation

– a direct consequence of assuming spin-$\frac{1}{2}$ quarks

- taking instead of eq.(156) the scalar analogue eq.(107)
  – gives a different relation

- the Callan-Gross relation fits the experiment quite well

$\Rightarrow$ quarks have spin-$\frac{1}{2}$

The quark parton model

- using the quantum numbers of the quarks (Table 1) we can write

$$F_2^{ep}(x) = x\left[\tfrac{4}{9}[u(x)+\bar{u}(x)] + \tfrac{1}{9}[d(x)+\bar{d}(x)+s(x)+\bar{s}(x)] + \dots\right] \tag{170}$$

- using iso-spin invariance
  - the neutron is like the proton, just up- and down quarks interchanged

$$u^p(x) = d^n(x) \equiv u(x) \qquad d^p(x) = u^n(x) \equiv d(x) \tag{171}$$

  we have

$$F_2^{en}(x) = x\left[\tfrac{4}{9}[d(x)+\bar{d}(x)] + \tfrac{1}{9}[u(x)+\bar{u}(x)+s(x)+\bar{s}(x)] + \dots\right] \tag{172}$$

- from strangeness and charges of proton and neutron we get sumrules

$$\int_0^1 dx[s-\bar{s}] = 0 \quad \int_0^1 dx[\tfrac{2}{3}(u-\bar{u}) - \tfrac{1}{3}(d-\bar{d})] = 1 \quad \int_0^1 dx[\tfrac{2}{3}(d-\bar{d}) - \tfrac{1}{3}(u-\bar{u})] = 0 \tag{173}$$

  - these are fulfilled by taking valence- and sea quarks:

$$u = u_V + q_S \quad d = d_V + q_S \quad \bar{u} = \bar{d} = s = \bar{s} = q_S \tag{174}$$

- and a momentum sumrule $\displaystyle\int_0^1 dx\, x[u+\bar{u}+d+\bar{d}+s+\bar{s}] = 1-\epsilon$ (175)

9. Deep Inelastic Electro-Nucleon Scattering and the Quark Parton Model

The Drell-Yan process

- the same probability distributions (parton distribution functions) are used for proton-(anti)proton collisions
  - the easiest process goes via a virtual photon: Drell-Yan process

$$p + p \to \mu^+\mu^- + X \tag{176}$$

- in the CM-frame we have (ignoring all masses)

$$p_1^\mu = (P, 0, 0, P) \qquad p_2^\mu = (P, 0, 0, -P) \quad \text{and} \quad s = 4P^2 \tag{177}$$

  - the momenta of the participating quarks are

$$p_{q_1}^\mu = x_1(P, 0, 0, P) \qquad p_{q_2}^\mu = x_2(P, 0, 0, -P) \tag{178}$$

  - the photon momentum is

$$q^\mu = p_{q_1}^\mu + p_{q_2}^\mu = P(x_1 + x_2, 0, 0, x_1 - x_2) \quad \Rightarrow \quad q^2 = 4P^2 x_1 x_2 \tag{179}$$

- the QED process $q\bar{q} \to \mu^+\mu^-$ is the same as $e^+e^- \to \mu^+\mu^-$

  ⇒ take the same cross section (Problem 8.19) . . . and next slide
  - the cross section will be proportional to the parton distribution functions
  - and the interval in x_1 and x_2 – with both combinations $q(x_1)$ and $q(x_2)$

$$d^2\sigma_{(pp\to\mu^+\mu^-+X)} = \sum_a \sigma_{(q_a\bar{q}_a\to\mu^+\mu^-+X)}[q_a(x_1)\bar{q}_a(x_2) + q_a(x_2)\bar{q}_a(x_1)]\, dx_1\, dx_2 \tag{180}$$

The Drell-Yan process

- massless fermion CM cross section: from eq.(28), eq.(36) and eq.(139)

$$\langle\sigma\rangle = \int \frac{\langle|\mathcal{M}|^2\rangle}{(8\pi)^2 s} d\Omega = \int \frac{\frac{e^4}{s^2} L_{\mu\nu}(k,k') L^{\mu\nu}(p,p')}{(8\pi)^2 s} 2\pi d(\cos\theta) \tag{181}$$

- in the CM frame for massless fermions we have

$$2p.k = p^2 + k^2 - (p-k)^2 = -\tfrac{s}{2}(1+\cos\theta) \tag{182}$$
$$2p.k' = p^2 - k'^2 - (p-k')^2 = -\tfrac{s}{2}(1+\cos\theta) \tag{183}$$

  – and analogous to eq.(140)

$$\begin{aligned} L_{\mu\nu}L^{\mu\nu} &= 2[(2p.k)^2 + (2p.k')^2] = 2[\tfrac{s^2}{4}(1-\cos\theta)^2 + \tfrac{s^2}{4}(1+\cos\theta)^2] \\ &= s^2(1+\cos^2\theta) \end{aligned} \tag{184}$$

  – giving the cross section

$$\langle\sigma\rangle = \frac{\pi\alpha^2}{2s}\int (1+\cos^2\theta) d(\cos\theta) = \frac{\pi\alpha^2}{2s}(2+\tfrac{2}{3}) = \frac{4\pi\alpha^2}{3s} \tag{185}$$

9. Deep Inelastic Electro-Nucleon Scattering and the Quark Parton Model

e^+e^- annihilation into hadrons

- assuming the dominance of the one-photon process
  - it should be similar to the Drell-Yan process, just reversed

$$\langle\sigma_{(e^+e^-\to q_a\bar{q}_a)}\rangle = \frac{4\pi\alpha^2}{3s}e_a^2 \qquad (186)$$

  - going into hadrons (i.e. summing over all possible quarks) gives

$$\langle\sigma_{(e^+e^-\to \text{hadrons})}\rangle = \frac{4\pi\alpha^2}{3s}\sum_a e_a^2 \qquad (187)$$

- taking the ratio to muon production:

$$R = \frac{\langle\sigma_{(e^+e^-\to \text{hadrons})}\rangle}{\langle\sigma_{(e^+e^-\to\mu^+\mu^-)}\rangle} = \sum_a e_a^2 \qquad (188)$$

$\Rightarrow$ prediction for R depends on the energy:

$$R \;=\; 3\cdot[(\tfrac{2}{3})^2+(-\tfrac{1}{3})^2+(-\tfrac{1}{3})^2+\Theta(E-m_c)(\tfrac{2}{3})^2+\Theta(E-m_b)(-\tfrac{1}{3})^2] = 2\,\Big|\,\tfrac{10}{3}\,\Big|\,\tfrac{11}{3} \qquad (189)$$

- the factor 3 describes the number of colors in QCD
  - essential for reproducing the experimental values $\Rightarrow$ proof for 3 colors