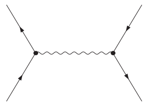
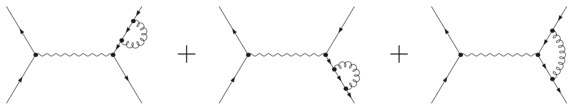
QCD II: Asymptotic freedom, the renormalization group and scaling violations in deep inelastic scattering

Going to the next-to-leading order of Drell-Yan production with the tree level matrix element

$F_\gamma =$ 

we have to include the 1-loop QCD corrections with the matrix element

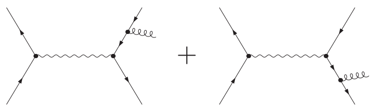
$F_{g,v} =$ 

to get the squared matrix element

$$|F_\gamma + F_{g,v}|^2 \quad = \quad |F_\gamma|^2 + 2\mathrm{Re}[F_\gamma^\dagger F_{g,v}] + |F_{g,v}|^2 \sim |F_\gamma|^2 + 2\mathrm{Re}[F_\gamma^\dagger F_{g,v}] = \mathcal{O}(\alpha_s^0) + \mathcal{O}(\alpha_s^1) \ . \tag{1}$$

The one loop part $F_{g,v}$ will diverge for some loop momentum ...

But adding the cross section of

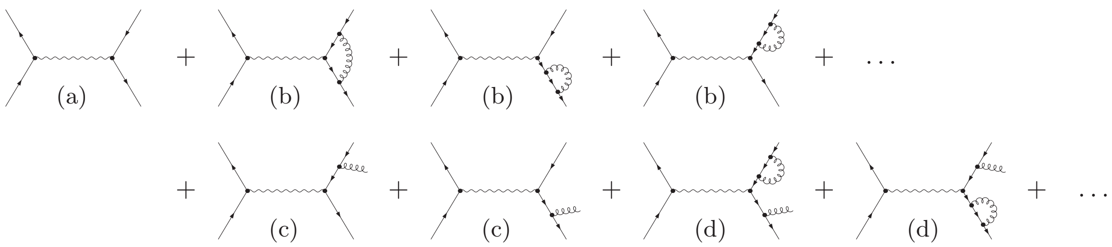
$F_{g,r} =$ 

for momenta of the gluon, that cannot be seperated from one of the quarks, will give the same divergence, but with an opposite sgine and hence cancel the divergence in $F_{g,v}$: soft and collinear gluons.

This cancellation is a general result stated in the Bloch-Nordsieck theorem (1937).

Another theorem, by Kinoshita (1962) and Lee & Nauenburg (1964), states, that mass singularities (i.e. singluarities occuring for a mass parameter going to zero) are absent, if all indistinguishable mass-degenerate states are added up. Then only singularities coming from the inital lines can remain. These give in deep inelastic scattering (DIS) the scaling violations.

Going back to $e^+e^- \to q\bar{q} + ng$, i.e.:



For the cross section we get a term proportional α^2 from the tree level (a), terms proportional $\alpha^2\alpha_s$ from (b) and (c), and terms proportional $\alpha^2\alpha_s^2$ from (d). But these higher order terms come with a factor

$$\sigma_{\mathrm{pt}} \left[-b\frac{\alpha_s^2}{\pi} \ln\left(\frac{s}{\mu^2}\right) \right] \ ,$$

where μ^2 is some mass scale. In the Standard Model $b = \frac{33-2N_f}{12\pi}$ with N_f counting the active flavors. So the ratio of the terms of $\mathcal{O}(\alpha^2\alpha_s)$ and of $\mathcal{O}(\alpha^2\alpha_s^2)$ is

$$-b\alpha_s \ln\left(\frac{s}{\mu^2}\right) \sim 1 \qquad \text{for } N_f = 5\,, \quad \alpha_s \sim 0.4\,, \quad \mu \sim 1\,GeV\,, \quad s \sim (10\,GeV)^2 \ .$$

So the perturbation series "fails".

Summing these terms anyway, we get

$$\sigma_{\mathrm{pt}} \left[1 + \frac{\alpha_s}{\pi}\left(1 - b\alpha_s \ln\left(\frac{s}{\mu^2}\right)\right)\right] =: \sigma_{\mathrm{pt}} \left[1 + \frac{\alpha_s(\mu^2)}{\pi}\right] \ ,$$

where we introduce the effective coupling constant $\alpha_s(\mu^2)$.

Calculating the next order, $\mathcal{O}(\alpha^2\alpha_s^3)$, one gets additional terms

$$+\sigma_{\mathrm{pt}}\frac{\alpha_s}{\pi}\left[b\alpha_s \ln\left(\frac{s}{\mu^2}\right)\right]^2 \ ,$$

which gives the impression, that we have here a series like $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ Assuming, that this is true, we get for the "resummed" cross section

$$\sigma_{\rm pt}\left[1 + \frac{\alpha_s/\pi}{1 + b\alpha_s \ln(s/\mu^2)}\right] = \sigma_{\rm pt}\left[1 + \frac{\alpha_s(\mu^2)}{\pi}\right] \quad .$$

The large logarithms are now in the denominator, which reduce now the effective coupling $\alpha_s(\mu^2)$, when the energy grows $\Rightarrow$ asymptotic freedom.

Renormalization group — dealing with large logarithms

We will consider only QED, since it is much simpler than QCD, but has the same technical features. The renormalization of the charge was written

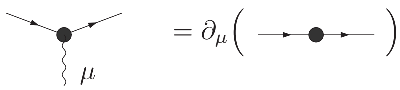
$$e = e_0 \left(\frac{Z_2}{Z_1}\right) Z_3^{1/2} \quad .$$

Due to the Ward identity we had $Z_1 = Z_2$.

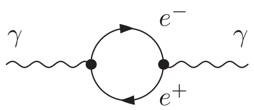
As a reminder: the renormalization constants were defined as the terms in the Lagranigan, proportional to

$$\begin{aligned} Z_1 &\quad : \quad Z_1 \cdot e\bar{\hat{\Psi}}\gamma^\mu \hat{\Psi}\,\hat{A}_\mu \\ Z_2 &\quad : \quad Z_2 \cdot \bar{\hat{\Psi}}\gamma^\mu i\partial_\mu \hat{\Psi} \\ Z_3 &\quad : \quad Z_3 \cdot \hat{F}_{\mu\nu}\hat{F}^{\mu\nu} \end{aligned}$$

and the Ward identity from eq.(11.6) can be written graphically as



From



we got $Z_3^{[2]} = 1 + \Pi_\gamma^{[2]}(0)$ with

$$\Pi_\gamma^{[2]}(q^2) = 8e^2\, i \int_0^1 dx \int \frac{d^4k'}{(2\pi)^4} \frac{x(1-x)}{(k'^2 - \Delta_\gamma + i\epsilon)^2} \qquad \text{where } \Delta_\gamma = m_e^2 - x(1-x)q^2 \ , \quad \text{and } q^2 < 0 \ . \qquad (2)$$

Regularizing with a cutoff for $|k'| < \Lambda$ we got

$$\Pi_\gamma^{[2]}(q^2) = -\frac{e^2}{\pi^2}\int_0^1 dx\;\; x(1-x)\left[\ln\frac{\Lambda + \sqrt{\Lambda^2 + \Delta_\gamma}}{\sqrt{\Delta_\gamma}} - \frac{\Lambda}{\sqrt{\Lambda^2+\Delta_\gamma}}\right] \quad .$$

Setting $q^2 = 0$ and expanding in $\frac{1}{\Lambda}$ we have $\Delta_\gamma = m_e^2$, $\sqrt{\Lambda^2 + \Delta_\gamma} = \Lambda\sqrt{1 + \frac{m_e^2}{\Lambda^2}} \sim \Lambda + \frac{m_e^2}{2\Lambda}$

$$\begin{aligned} \Pi_\gamma^{[2]}(0) &= -\frac{e^2}{\pi^2}\int_0^1 dx\;\; x(1-x)\left[\ln\frac{\Lambda + \sqrt{\Lambda^2 + m_e^2}}{m_e} - \frac{\Lambda}{\sqrt{\Lambda^2 + m_e^2}}\right] \\ &= -\frac{e^2}{\pi^2}\left[\ln\left(\frac{\Lambda}{m_e}\left[1 + \sqrt{1 + \frac{m_e^2}{\Lambda^2}}\right]\right) - \left[1 + \frac{m_e^2}{\Lambda^2}\right]^{-1/2}\right]\int_0^1 dx\;\; x(1-x) \\ &\sim -\frac{4\alpha}{\pi}\left[\ln\frac{\Lambda}{m_e} + \ln\left(1 + 1 + \frac{1}{2}\frac{m_e^2}{\Lambda^2}\right) - \left[1 - \frac{1}{2}\frac{m_e^2}{\Lambda^2}\right]\right]\cdot\frac{1}{6} \\ &\sim -\frac{2\alpha}{3\pi}\ln\frac{\Lambda}{m_e} \ , \end{aligned}$$

and

$$\sqrt{Z_3^{[2]}} = 1 + \tfrac{1}{2}\Pi_\gamma^{[2]}(0) = 1 - \frac{\alpha}{3\pi}\ln\frac{\Lambda}{m_e} \quad .$$

The renormalized selfenergy

$$\bar{\Pi}_\gamma^{[2]}(q^2) := \Pi_\gamma^{[2]}(q^2) - \Pi_\gamma^{[2]}(0) \tag{3}$$

gets rid of the cutoff dependence. So

$$\bar{\Pi}_\gamma^{[2]}(q^2) = -\frac{e^2}{\pi^2}\int_0^1 dx\;\, x(1-x)\ln\left[\frac{m_e^2}{m_e^2 - x(1-x)q^2}\right] \quad \stackrel{\text{large } q^2}{\longrightarrow} \quad \frac{\alpha}{3\pi}\ln\frac{|q^2|}{m_e^2} \quad .$$

This is how the scale m_e enters via the renormaliation constant Z_3.

Choosing another point for the definition of the renormalization procedure, for instance $q^2 = -\mu^2$, we just get a different value for the charge, defined at a different point. This scale is arbitrary, but the physical result has to stay the same!

Changing the renormalization scale: with changing μ^2 we can change the large logarithm $\ln\frac{-q^2}{\mu^2}$. For the counterterm $Z_3^{[2]}(\mu) = 1 + \Pi_\gamma^{[2]}(q^2 = -\mu^2)$, defined at μ^2, we get

$$\sqrt{Z_3} = 1 + \tfrac{1}{2}\Pi_\gamma^{[2]}(q^2 = -\mu^2) = 1 - \frac{\alpha}{3\pi}\ln\frac{\Lambda}{\mu} \quad .$$

Then the renormalized selfenergy becomes

$$\bar{\Pi}_\gamma^{[2]}(q^2, \mu^2) := \Pi_\gamma^{[2]}(q^2) - \Pi_\gamma^{[2]}(q^2 = -\mu^2) = -\frac{e^2}{2\pi^2}\int_0^1 dx\;\, x(1-x)\ln\left[\frac{m_e^2 + x(1-x)\mu^2}{m_e^2 - x(1-x)q^2}\right] \quad .$$

For $m_e \ll |q^2|, \mu^2$, the logarithm goes to $\ln\frac{|q^2|}{\mu^2}$, which is small for $|q^2| \sim \mu^2$. Therefore also the "large logs" $-b\alpha_s \ln\frac{|q^2|}{\mu^2}$ are small! But we have to define now the coupling also at μ^2:

$$e_{(\mu)} = \sqrt{Z_3^{[2]}(\mu)} \cdot e_0 = (1 - \frac{\alpha}{3\pi}\ln\frac{\Lambda}{\mu})e_0$$

and not at $\mu = 0$:

$$e = \sqrt{Z_3^{[2]}(0)} \cdot e_0 = (1 - \frac{\alpha}{3\pi}\ln\frac{\Lambda}{m_e})e_0 \quad .$$

So

$$e_{(\mu)} = \frac{1 - \frac{\alpha}{3\pi}\ln\frac{\Lambda}{\mu}}{1 - \frac{\alpha}{3\pi}\ln\frac{\Lambda}{m_e}} e \approx \left(1 + \frac{\alpha}{3\pi}\ln\frac{\mu}{m_e}\right) e$$

We should change the scale from m_e to a large μ in many steps, in order to stay in the valid regime for the perturbation expansion. We can regard $e_{\mu'}$ as a continuous function of the scales μ and μ'. The only scales in the problem are μ, μ', and m_e. So we *have to have*

$$e_{\mu'} = E(e_\mu, \tfrac{\mu'}{\mu}, \tfrac{m_e}{\mu}) \quad .$$

Differentiating logarithmically with respect to μ' gives

$$\mu'\frac{\mathrm{d}}{\mathrm{d}\mu'}e_{\mu'} = \mu'\frac{\partial}{\partial\mu'}E(e_\mu, \tfrac{\mu'}{\mu}, \tfrac{m_e}{\mu}) \quad ,$$

letting $\mu' \to \mu$ gives

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}e_{\mu'} = \left[\frac{\partial}{\partial z}E(e_\mu, z, \tfrac{m_e}{\mu})\right]_{z=1} := \beta(e_\mu, \tfrac{m_e}{\mu}) \quad \stackrel{m_e \ll \mu}{=} \quad \beta(e_\mu, 0) = \beta(e_\mu) \quad ,$$

the Callen Symanzik equation.

The bare coupling

$$e_0 = \left(1 - \frac{\alpha}{3\pi}\ln\frac{\Lambda}{\mu}\right)^{-1} e_{(\mu)} \sim \left(1 + \frac{\alpha}{3\pi}\ln\frac{\Lambda}{\mu}\right) e_{(\mu)} \sim \left(1 + \frac{e_{(\mu)}^2}{12\pi^2}\ln\frac{\Lambda}{\mu}\right) e_{(\mu)} = e_{(\mu)} + \frac{e_{(\mu)}^3}{12\pi^2}\ln\frac{\Lambda}{\mu}$$

is independent of μ: differentiating with respect to μ for fixed e_0 gives

$$0 \quad = \quad \frac{\mathrm{d}e_{(\mu)}}{\mathrm{d}\mu} + 3\frac{\mathrm{d}e_{(\mu)}}{\mathrm{d}\mu}\frac{e_{(\mu)}^2}{12\pi^2}\ln\frac{\Lambda}{\mu} + \frac{e_{(\mu)}^3}{12\pi^2}(\frac{\Lambda}{\mu})^{-1}(-\frac{\Lambda}{\mu^2}) = \frac{\mathrm{d}e_{(\mu)}}{\mathrm{d}\mu}\left(1 + \frac{e_{(\mu)}^2}{4\pi^2}\ln\frac{\Lambda}{\mu}\right) - \frac{1}{\mu}\frac{e_{(\mu)}^3}{12\pi^2}$$

or

$$\mu\frac{\mathrm{d}e_{(\mu)}}{\mathrm{d}\mu} \quad = \quad \frac{e_{(\mu)}^3}{12\pi^2}\left(1 + \frac{e_{(\mu)}^2}{4\pi^2}\ln\frac{\Lambda}{\mu}\right)^{-1} \stackrel{\text{to second order}}{=} \frac{e_{(\mu)}^3}{12\pi^2} = \beta^{[2]}(e_{(\mu)}) \ .$$

This can easily be solved by rewriting the equation as

$$\frac{\mathrm{d}e_{(\mu)}}{e_{(\mu)}^3} = \frac{1}{12\pi^2}\frac{\mathrm{d}\mu}{\mu}$$

and integrating between the scales μ_1 and μ_2:

$$\int_{e_{(\mu_1)}}^{e_{(\mu_2)}} \frac{\mathrm{d}e_{(\mu)}}{e_{(\mu)}^3} = \left[-\frac{1}{2}\frac{1}{e_{(\mu)}^2}\right]_{e_{(\mu_1)}}^{e_{(\mu_2)}} = -\frac{1}{2}\left[\frac{1}{e_{(\mu_2)}^2} - \frac{1}{e_{(\mu_1)}^2}\right] = \int_{\mu_1}^{\mu_2}\frac{1}{12\pi^2}\frac{\mathrm{d}\mu}{\mu} = \left[\frac{1}{12\pi^2}\ln\mu\right]_{\mu_1}^{\mu_2} = \frac{1}{12\pi^2}\ln\frac{\mu_2}{\mu_1}$$

and with $\mu_2 = \mu$ and $\mu_1 = M$ we have

$$\frac{1}{e_{(\mu)}^2} = \frac{1}{e_{(M)}^2} - \frac{1}{12\pi^2}\ln\frac{\mu^2}{M^1} = \frac{1 - \frac{e_{(M)}^2}{12\pi^2}\ln\frac{\mu^2}{M^1}}{e_{(M)}^2} \qquad \text{or} \quad e_{(\mu)}^2 = \frac{e_{(M)}^2}{1 - \frac{e_{(M)}^2}{12\pi^2}\ln\frac{\mu^2}{M^1}}$$

or with $\alpha = \frac{e^2}{4\pi}$

$$\alpha_{(\mu)} = \frac{\alpha_{(M)}}{1 - \frac{\alpha_{(M)}}{3\pi}\ln\frac{\mu^2}{M^1}}$$

and the possibly large logarithm is now safely in the denominator.

The general solution for the coupling constant can be obtained from the equation

$$\ln\frac{\mu}{M} = \int_{e_{(M)}}^{e_{(\mu)}}\frac{\mathrm{d}e}{\beta(e)} \ .$$

Since physical observables *cannot* depend on te choice of the renormalization scale μ, we can express *dimensionless* quantities in terms of dimensionless quantities: $\frac{m^2}{\mu^2}$, $\frac{s}{\mu^2}$, …etc. So

$$S := \frac{\sigma^{\text{1lopp}}}{\sigma^{\text{point}}} = S\left(\frac{|q^2|}{\mu^2}, \frac{m^2}{\mu^2}; e_{(\mu)}\right) \stackrel{\text{ignoring masses}}{=} S\left(\frac{|q^2|}{\mu^2}; e_{(\mu)}\right) \ ,$$

and

$$0 \quad = \quad \mu\frac{\mathrm{d}S}{\mathrm{d}\mu} = \left(\mu\frac{\partial}{\partial\mu}\bigg|_{e_{(\mu)}} + \mu\frac{\mathrm{d}e_{(\mu)}}{\mathrm{d}\mu}\bigg|_{e_0}\frac{\partial}{\partial e_{(\mu)}}\right) S\left(\tfrac{|q^2|}{\mu^2}; e_{(\mu)}\right) = \left(\mu\frac{\partial}{\partial\mu}\bigg|_{e_{(\mu)}} + \beta(e_{(\mu)})\frac{\partial}{\partial e_{(\mu)}}\right) S\left(\tfrac{|q^2|}{\mu^2}; e_{(\mu)}\right) \ ,$$

which describes the **Renormalization Group Equation (RGE)** for S. The normal definition uses $\alpha = \frac{e^2}{4\pi}$ and μ^2 instead of e and μ:

$$\left(\mu^2\frac{\partial}{\partial\mu^2}\bigg|_{\alpha_{(\mu)}} + \beta(\alpha_{(\mu)})\frac{\partial}{\partial\alpha_{(\mu)}}\right) S\left(\tfrac{|q^2|}{\mu^2}; \alpha_{(\mu)}\right) = 0 \ ,$$

where $\beta(\alpha) := \mu^2\frac{\partial\alpha_{(\mu^2)}}{\partial\mu^2}$, which at 1 loop has the value $\beta^{[2]}(\alpha) := \frac{\alpha^2}{3\pi}$.

When we change to the logarithmic variable

$$t := \ln \frac{|q^2|}{\mu^2} \qquad \text{or} \qquad \frac{|q^2|}{\mu^2} = e^t$$

we get $\mu^2 = |q^2| e^{-t}$ and

$$\mu^2 \frac{\partial}{\partial \mu^2} = \mu^2 \frac{\partial t}{\partial \mu^2} \frac{\partial}{\partial t} = \mu^2 \frac{\mu^2}{|q^2|} (-\frac{|q^2|}{(\mu^2)^2}) \frac{\partial}{\partial t} = -\frac{\partial}{\partial t}$$

and

$$\left(-\frac{\partial}{\partial t}\bigg|_{\alpha_{(\mu)}} + \beta(\alpha_{(\mu)}) \frac{\partial}{\partial \alpha_{(\mu)}} \right) S\left(e^t; \alpha_{(\mu)}\right) = 0 \ , \tag{4}$$

which can be solved exactly by introducing $\alpha(|q^2|)$ via the definition

$$t = \int_{\alpha_{(\mu)}}^{\alpha(|q^2|)} \frac{\mathrm{d}\alpha}{\beta(\alpha)} \ . \tag{5}$$

Proof. The rule for differentiating an integral with respect to a limit is

$$\frac{\partial}{\partial a} \int^{f(a)} g(x) \mathrm{d}x = g(f(a)) \frac{\partial f}{\partial a} \ .$$

Differentiating eq.(5) with respect to $t = \ln \frac{|q^2|}{\mu^2}$ for fixed $\alpha_{(\mu)}$ gives

$$1 = \frac{1}{\beta(\alpha(|q^2|))} \frac{\partial \alpha(|q^2|)}{\partial t} \qquad \text{or} \quad \beta(\alpha(|q^2|)) = \frac{\partial \alpha(|q^2|)}{\partial t} \ .$$

Differentiating eq.(5) with respect to $\alpha_{(\mu)}$ for fixed t gives

$$0 = \frac{1}{\beta(\alpha(|q^2|))} \frac{\partial \alpha(|q^2|)}{\partial \alpha_{(\mu)}} - \frac{1}{\beta(\alpha_{(\mu)})} \qquad \text{or} \quad \beta(\alpha_{(\mu)}) \frac{\partial \alpha(|q^2|)}{\partial \alpha_{(\mu)}} - \beta(\alpha(|q^2|)) = 0 \ .$$

Both together give

$$\beta(\alpha_{(\mu)}) \frac{\partial \alpha(|q^2|)}{\partial \alpha_{(\mu)}} - \frac{\partial \alpha(|q^2|)}{\partial t} = \left[-\frac{\partial}{\partial t} \beta(\alpha_{(\mu)}) \frac{\partial}{\partial \alpha_{(\mu)}} \right] \alpha(|q^2|) = 0 \ .$$

This makes $S(e^t, \alpha_{(\mu)}) = S(1, \alpha(|q^2|))$ a solution of eq.(4):

$$\begin{aligned}
& \left(-\frac{\partial}{\partial t} + \beta(\alpha_{(\mu)}) \frac{\partial}{\partial \alpha_{(\mu)}} \right) S(1, \alpha(|q^2|)) \\
= & \quad -\frac{\partial \alpha(|q^2|))}{\partial t} \frac{\partial S(1, \alpha(|q^2|))}{\partial \alpha(|q^2|)} + \beta(\alpha_{(\mu)}) \frac{\partial \alpha(|q^2|))}{\partial \alpha_{(\mu)}} \frac{\partial S(1, \alpha(|q^2|))}{\partial \alpha(|q^2|)} \\
= & \quad \left(\left[-\frac{\partial \alpha(|q^2|))}{\partial t} + \beta(\alpha_{(\mu)}) \frac{\partial}{\partial \alpha_{(\mu)}} \right] \alpha(|q^2|)) \right) \frac{\partial S(1, \alpha(|q^2|))}{\partial \alpha(|q^2|)} = 0 \ .
\end{aligned} \tag{6}$$

That means, that $S(\frac{|q^2|}{\mu^2}, \alpha_{(\mu)})$ depends on $|q^2|$ **only** through the dependence of $\alpha(|q^2|)$ on $|q^2|$. This allows predictions of the leading energy dependence of S.

Integrating the β-function (to one loop order) we got

$$\alpha(|q^2|) = \frac{\alpha_{(\mu)}}{1 - \frac{\alpha_{(\mu)}}{3\pi} \ln \frac{|q^2|}{\mu^2}} = \frac{\alpha_{(\mu)}}{1 - \frac{\alpha_{(\mu)}}{3\pi} t} \ ,$$

from eq.(15.30) with the replacements $\mu^2 \to |q^2|$ and $M \to \mu$.

Calculating perturbative corrections we can write the result as the perturbation expansion in the coupling constant:

$$S(1, \alpha_{(\mu)}) = S_1 \alpha_{(\mu)} + S_2 \alpha_{(\mu)}^2 + S_3 \alpha_{(\mu)}^3 + \dots$$

we get $S(1, \alpha(|q^2|))$

$$
\begin{aligned}
&= S_1\alpha(|q^2|) + S_2[\alpha(|q^2|)]^2 + S_3[\alpha(|q^2|)]^3 + \dots \\
&= S_1\alpha_{(\mu)}\left[1 + \tfrac{\alpha_{(\mu)}t}{3\pi} + \left(\tfrac{\alpha_{(\mu)}t}{3\pi}\right)^2 + \left(\tfrac{\alpha_{(\mu)}t}{3\pi}\right)^3 + \dots\right] + S_2\alpha^2_{(\mu)}\left[1 + \tfrac{\alpha_{(\mu)}t}{3\pi} + \left(\tfrac{\alpha_{(\mu)}t}{3\pi}\right)^2 + \dots\right]^2 \\
&\quad + S_3\alpha^3_{(\mu)}\left[1 + \tfrac{\alpha_{(\mu)}t}{3\pi} + \dots\right]^3 + \dots \\
&= S_1\alpha_{(\mu)}\left[1 + \tfrac{\alpha_{(\mu)}t}{3\pi} + \left(\tfrac{\alpha_{(\mu)}t}{3\pi}\right)^2 + \left(\tfrac{\alpha_{(\mu)}t}{3\pi}\right)^3 + \dots\right] + S_2\alpha^2_{(\mu)}\left[1 + 2\tfrac{\alpha_{(\mu)}t}{3\pi} + 3\left(\tfrac{\alpha_{(\mu)}t}{3\pi}\right)^2 + \dots\right] \\
&\quad + S_3\alpha^3_{(\mu)}\left[1 + 3\tfrac{\alpha_{(\mu)}t}{3\pi} + \dots\right] + \dots \\
&= \alpha_{(\mu)}S_1 + \alpha^2_{(\mu)}(\tfrac{t}{3\pi}S_1 + S_2) + \alpha^3_{(\mu)}((\tfrac{t}{3\pi})^2)S_1 + 2\tfrac{t}{3\pi}S_2 + S_3 + \dots
\end{aligned}
$$

So using $\alpha(|q^2|)$ instead of $\alpha_{(\mu)}$ sums up the leading logarithmic terms $\frac{t}{3\pi}$.

$\Rightarrow$ Now we should apply the same logic to QCD.

Asymptotic freedom

The term b from eq.(15.4) gvies the second order term for the β-function of QCD:

$$
\beta_s = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}\bigg|_{\text{fixed bare } \alpha_s} \qquad \text{with} \quad \beta_s^{[2]} = -b\alpha_s^2 \qquad \text{where} \quad b = \frac{33 - 2N_f}{12\pi} \quad .
$$

For $N_f \leq 16$ we have $b > 0$, whereas the corresponding value for QED is $b_{\text{QED}} = -\frac{1}{3\pi} < 0$, i.e.

$$
\mu^2 \frac{\partial \alpha}{\partial \mu^2} = \beta = -b_{\text{QED}}\alpha^2 \quad .
$$

So from the analogy

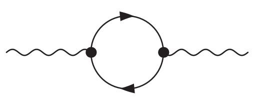
$$
\alpha(|q^2|) = \frac{\alpha_{(\mu)}}{1 - \frac{\alpha_{(\mu)}}{3\pi}\ln\frac{|q^2|}{\mu^2}} = \frac{\alpha_{(\mu)}}{1 + b_{\text{QED}}\alpha_{(\mu)}\ln\frac{|q^2|}{\mu^2}}
$$

we get

$$
\alpha_s(|q^2|) = \frac{\alpha_{(\mu)}}{1 + b\alpha_s(\mu)\ln\frac{|q^2|}{\mu^2}} \quad .
$$

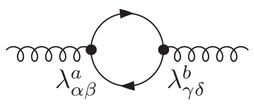
So $\alpha_s(|q^2|)$ runs in the opposite direction compared to $\alpha_{\text{QED}}(|q^2|)$: α_s gets *smaller* with *higher* energies!

In QED we had only the "vacuum sceening"

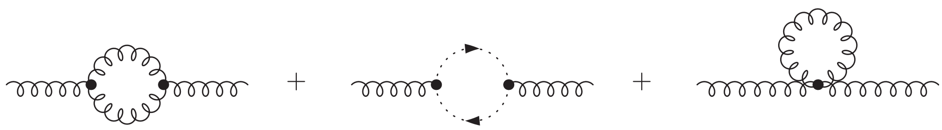


which gives for the single electron $b_{\text{QED}} = -\frac{1}{3\pi}$ or for N_f flavours $-\frac{N_f}{3\pi}$.

In QCD we get



with the colorfactors $\sum_{\alpha,\beta,\gamma,\delta}(\frac{1}{2}\lambda^a)_{\alpha\beta}\delta_{\beta\gamma}(\frac{1}{2}\lambda^b)_{\gamma\delta}\delta_{\delta\alpha} = \frac{1}{2}\delta^{ab}$. So the $\frac{1}{3\pi}$ becomes $\frac{1}{6\pi} = \frac{2}{12\pi}$ for each flavor. But



give the other contribution of $+\frac{33}{12\pi}$.

From Peskin & Schroeder, Chapter 16, eq.(16.85):

$$
\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right)
$$

for $SU(N)$ we have the values $C_2(G) = N$ and $C(r) = \frac{1}{2}$, so

$$\beta_s(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N - \frac{4}{3}n_f\frac{1}{2}\right) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N - \frac{4}{3}n_f\frac{1}{2}\right)$$

and

$$\beta_s(\alpha_s) = -\frac{\alpha_s^2}{4\pi}\left(\frac{11}{3}3 - \frac{2}{3}n_f\right) = -\frac{33-2n_f}{12\pi}\alpha_s^2 \ .$$

The running coupling can also be rewritten with the definition

$$\ln\frac{\mu^2}{\Lambda^2_{\mathrm{QCD}}} = -\int_{\alpha_s(\mu^2)}^{\infty}\frac{\mathrm{d}\,\alpha_s}{\beta_s} \sim -\int_{\alpha_s(\mu^2)}^{\infty}\frac{\mathrm{d}\,\alpha_s}{\beta_s^{[2]}} = \int_{\alpha_s(\mu^2)}^{\infty}\frac{\mathrm{d}\,\alpha_s}{b\alpha_s^2} = \left[-\frac{1}{b\alpha_s}\right]_{\alpha_s(\mu^2)}^{\infty} = \frac{1}{b\alpha_s(\mu^2)} \ ,$$

or with $|q^2| = \mu^2$

$$\ln\Lambda^2_{\mathrm{QCD}} = \ln|q^2| - \frac{1}{b\alpha_s(|q^2|)} \ ,$$

which picks a specific scale μ^2. Then

$$\alpha_s(|q^2|) = \frac{1}{b\ln\frac{|q^2|}{\Lambda^2_{\mathrm{QCD}}}} \ ;$$

the extraction of Λ_{QCD} is complicated, but gives typically $\Lambda_{\mathrm{QCD}} \sim 200\,\mathrm{MeV} \sim (\mathrm{fm})^{-1}$.

At two-loop the term for the β-function

$$\beta_s^{[3]} = -b\,b'\,\alpha_s^3 \sim \frac{153-19N_f}{24\pi^2}\alpha_s^3$$

involves terms of the type $\ln(\ln\frac{|q^2|}{\mu^2})$. Three-loop results exist, but people only use then numerical evaluations of $\alpha_s(\mu^2)$.

Anomalous dimension and running masses

Recall the relation between the bare coupling and the coupling at the scale μ:

$$e_0 = \frac{Z_1}{Z_2}\frac{e_{(\mu)}}{\sqrt{Z_3}}$$

If now $Z_1 \neq Z_2$ then

$$\ln e_0 = \ln Z_1 - \ln Z_2 - \tfrac{1}{2}\ln Z_3 + \ln e_{(\mu)} \ .$$

The logarithmic derivative $\frac{\mathrm{d}}{\mathrm{d}\ln\mu} = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}$ gives the equation

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln e_0 = 0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_1\bigg|_{e_0} - \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_2\bigg|_{e_0} - \frac{1}{2}\,\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_3\bigg|_{e_0}\frac{\mu}{e_{(\mu)}}\frac{\mathrm{d}e_{(\mu)}}{\mathrm{d}\mu}$$

or

$$\mu\frac{\mathrm{d}e_{(\mu)}}{\mathrm{d}\mu} = e_{(\mu)}(\gamma_3 + 2\gamma_2 - \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_1) \qquad \text{where} \quad \gamma_{2,3} = \frac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_{2,3} \ .$$

For $Z_1 \neq Z_2$ we need the fermion field renormalization and the vertex renormalization. Z_2 comes from the relation between the bare and the renormalized propagator

$$\langle\Omega|T\left(\bar{\hat{\Psi}}(x)\hat{\Psi}(0)\right)|\Omega\rangle = \frac{1}{Z_2}\langle\Omega|T\left(\bar{\hat{\Psi}}_0(x)\hat{\Psi}_0(0)\right)|\Omega\rangle$$

and

$$\tilde{S}'_F(q^2) = \int \mathrm{d}^4x\, e^{iq\cdot x}\,\langle\Omega|T\left(\bar{\hat{\Psi}}(x)\hat{\Psi}(0)\right)|\Omega\rangle \ .$$

The question we want to ask now is: how does $\tilde{S}'_F$ look for $|q^2| = -q^2 \gg m^2$? From the dimensionality we have $\tilde{S}'_F \propto \frac{1}{M}$ as we expect $\tilde{S}'_F$ to be like the propagator $\frac{i}{\not{q}-m}$. In order to get a dimensionless quantity we define

$$\tilde{R}'_F(\frac{|q^2|}{\mu^2}, \alpha_{(\mu)}) = \not{q}\tilde{S}'_F(q^2)$$

The we get the equation

$$\left[\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\bigg|_{\alpha_{(\mu)}} + \beta(\alpha_{(\mu)})\frac{\partial}{\partial\alpha_{(\mu)}} + \gamma_2(\alpha_{(\mu)})\right] \tilde{R}'_F(\frac{|q^2|}{\mu^2}, \alpha_{(\mu)}) = 0 \ ,$$

where γ_2 comes from the involvement of Z_2 in the definition of the renormalized propagator. For the fixed point $\beta = 0$ we get the simple solution

$$\tilde{R}'_F(\frac{|q^2|}{\mu^2}, \alpha_{(\mu)}) \propto (\mu^2)^{-\gamma_2(\alpha_{(\mu)})} \ .$$

And since $\tilde{R}'_F(\frac{|q^2|}{\mu^2}, \alpha_{(\mu)})$ only depends on q^2 through $\frac{|q^2|}{\mu^2}$, we get

$$\tilde{S}'_F(\frac{|q^2|}{\mu^2}, \alpha_{(\mu)}) \propto \frac{1}{\not{q}}\left(\frac{|q^2|}{\mu^2}\right)^{\gamma_2(\alpha_{(\mu)})} \ ,$$

which describes an "anomalous power law" dependence for the pro.
$\Rightarrow$ anomalous dimension of the fermion field, which means, that the full propagator of the interacting theory does not scale like the free propagator.

The full solution for $\beta \neq 0$ is

$$\tilde{R}'_F(\frac{|q^2|}{\mu^2}, \alpha_{(\mu)}) = \tilde{R}'_F(1, \alpha(|q^2|)) \cdot \exp\left\{\int_0^{t=\ln\frac{|q^2|}{\mu^2}} \mathrm{d}t' \, \gamma_2(\alpha(t'))\right\} \ .$$

Assuming a fixed point for the β-function, $\beta(\alpha^*) = 0$, we can expand the β-function in the vicinity of the fixed point $\beta(\alpha) = -B(\alpha - \alpha^*)$ and get for the definition of $\alpha(|q^2|)$:

$$\ln\frac{|q^2|}{\mu^2} = \int_{\alpha_{(\mu)}}^{\alpha(|q^2|)} \frac{\mathrm{d}\alpha}{-B(\alpha-\alpha^*)} = -\tfrac{1}{B}\left[\ln(\alpha-\alpha^*)\right]_{\alpha_{(\mu)}}^{\alpha(|q^2|)} = \tfrac{1}{B}\left[\ln(\alpha_{(\mu)}-\alpha^*) - \ln(\alpha(|q^2|)-\alpha^*)\right]$$

So

$$\ln(\alpha(|q^2|) - \alpha^*) = \ln(\alpha_{(\mu)} - \alpha^*) + B\ln\frac{\mu^2}{|q^2|}$$

and

$$\alpha(|q^2|) = \alpha^* + (\alpha_{(\mu)} - \alpha^*)\left(\frac{\mu^2}{|q^2|}\right)^B = \alpha^* + \mathrm{const}_{(\text{in } \mathrm{q}^2)}\left(\frac{\mu^2}{|q^2|}\right)^B \ .$$

Therefore $\alpha(|q^2|) \to \alpha^*$ for sufficiently large $|q^2|$. Then

$$\tilde{R}'_F(\frac{|q^2|}{\mu^2}, \alpha_{(\mu)}) \to \tilde{R}'_F(1, \alpha^*) \cdot \exp\left\{\int_0^t \mathrm{d}t' \, \gamma_2(\alpha^*)\right\} = \tilde{R}'_F(1, \alpha^*)e^{\gamma_2(\alpha^*)t} = \tilde{R}'_F(1, \alpha^*)\left(\frac{|q^2|}{\mu^2}\right)^{\gamma_2(\alpha^*)} \ .$$

One problem arises for gauge theories: S_F is **not** gauge invariant, and neither is Z_2 ...and hence also γ_2. One can still treat gauge invariant quantities this way, like the (total) scattering cross section, etc. ...

Quark masses. The pole prescritopn (i.e. "on-shell") **cannot** be used for particles that cannot be observed as free particles! ...But there is no need to use it either. We can treat the mass term $m\bar{\Psi}\Psi$ as a coupling. And since $\bar{\Psi}\Psi$ is a gauge invariant quantity we can use the RGE approach without any problem:

$$\left[\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s} + \sum_{i=\text{anomalous dimensions of } R}\gamma_i(\alpha_s) + \gamma_m(\alpha_s)m\frac{\partial}{\partial m}\right] \tilde{R}'_F(\tfrac{|q^2|}{\mu^2}, \alpha_s, \tfrac{m}{|q|}) = 0 \ ,$$

with $\beta(\alpha_s) = \frac{\partial}{\partial t}\alpha_s$, $\gamma_m(\alpha_s(|q^2|)) = \frac{\partial}{\partial t}\ln m(|q^2|)$, and $t = \frac{|q^2|}{\mu^2}$ and the definition for $m(|q^2|)$

$$m(|q^2|) = m(\mu^2)\cdot\exp\left\{\int_{\mu^2}^{|q^2|} \mathrm{d}\ln|q'^2|\,\gamma_2(\alpha_s(|q'^2|))\right\} \quad .$$

In QCD for one loop $\gamma_2(\alpha_s) = -\frac{\alpha_s}{\pi}$, so

$$m(|q^2|) = m(\mu^2)\cdot\left[\frac{\ln\left(\frac{\mu^2}{\Lambda^2}\right)}{\ln\left(\frac{|q^2|}{\Lambda^2}\right)}\right]^{\frac{1}{\pi b}=\frac{12}{33-2N_f}>0} \quad .$$

Therefore quark masses **decrease** logarithmically with increasing energy! So quark mass effects are not only suppressed by the energy as $\frac{m^2}{|q^2|}$, but by an additional suppression factor!

Technicalities: MS**,** $\overline{\text{MS}}$**, and differencies**

Dimensional regularization for eq.(2)

$$\Pi_\gamma^{[2]}(q^2) = 8e^2\,i\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{x(1-x)}{(k^2-\Delta_\gamma+i\epsilon)^2} \quad .$$

We recognize $\mathrm{d}^4k = \mathrm{d}k^0\mathrm{d}^3\vec{k}$. With the Wick rotation $k^0 \to ik^4$ we can avoid the poles in the propagator:

$$k^2-\Delta_\gamma+i\epsilon \to (ik^4)^2-\vec{k}^2-\Delta_\gamma+i\epsilon = -(\vec{k}^2+(k^4)^2+\Delta_\gamma)+i\epsilon \to -(\vec{k}_E^2+\Delta_\gamma) \quad ,$$

where we identify k^4 with a fourth Euclidean dimension. Next we split the integration over $\int_{-infty}^{infty}\mathrm{d}^4k_E$ into the integration of the solid angle $\mathrm{d}\Omega_E^3$ and the length ℓ with $\ell^2 = \vec{k}_E^2$. So we have

$$\Pi_\gamma^{[2]}(q^2) = 8e^2\,i\int_0^1 \mathrm{d}x\,x(1-x)\,i\int\frac{\mathrm{d}\Omega_E^3}{(2\pi)^4}\int_0^\infty \frac{\ell^3\mathrm{d}\ell}{(\ell^2+\Delta_\gamma)^2} \quad ,$$

where we now understand the integrals as dependent on the dimensionality D of the Euclidean space, that we integrate over:

$$\Pi_\gamma^{[2]}(q^2,D) = -8e^2\int_0^1 \mathrm{d}x\,x(1-x)\int\frac{\mathrm{d}\Omega_E^{D-1}}{(2\pi)^D}\int_0^\infty \frac{\ell^{D-1}\mathrm{d}\ell}{(\ell^2+\Delta_\gamma)^2} \quad .$$

Now we solve these integrals in dependence on D as a purely mathematical exercise.

With the substitution

$$y = \frac{\Delta_\gamma}{\ell^2+\Delta_\gamma} \qquad \text{so} \quad \ell^2 = \frac{\Delta_\gamma}{y}-\Delta_\gamma = \Delta_\gamma\frac{1-y}{y} \qquad \text{and} \quad \mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}\ell}\frac{\Delta_\gamma}{\ell^2+\Delta_\gamma}\mathrm{d}\ell = -\Delta_\gamma\frac{2\ell}{(\ell^2+\Delta_\gamma)^2}\mathrm{d}\ell$$

and the boundaries $y(0) = \frac{\Delta_\gamma}{0+\Delta_\gamma} = 1$ $y(\infty) = \frac{\Delta_\gamma}{\infty^2+\Delta_\gamma} = 0$ we have the ℓ integration part as

$$\begin{aligned}\int_0^\infty \frac{\ell^{D-1}\mathrm{d}\ell}{(\ell^2+\Delta_\gamma)^2} &= \frac{1}{2\Delta_\gamma}\int_0^\infty (\ell^2)^{D/2-1}\frac{\Delta_\gamma 2\ell\mathrm{d}\ell}{(\ell^2+\Delta_\gamma)^2} = \frac{1}{2\Delta_\gamma}\int_1^0\left(\Delta_\gamma\frac{1-y}{y}\right)^{D/2-1}(-\mathrm{d}y)\\ &= \frac{(\Delta_\gamma)^{D/2-2}}{2}\int_0^1 \mathrm{d}y\;\, y^{-D/2+1}(1-y)^{D/2-1} \quad ,\end{aligned}$$

which we can identify as the definition of the mathematical Beta-function

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 \mathrm{d}y\;\, y^{\alpha-1}(1-y)^{\beta-1} \quad .$$

So we can identify the exponents $\beta = D/2$ and $\alpha = (-D/2+1)+1 = 2-D/2$ and get

$$\int_0^\infty \frac{\ell^{D-1}\mathrm{d}\ell}{(\ell^2+\Delta_\gamma)^2} = \tfrac{1}{2}(\Delta_\gamma)^{\frac{D}{2}-2}\frac{\Gamma(2-\frac{D}{2})\Gamma(\frac{D}{2})}{\Gamma(2-\frac{D}{2}+\frac{D}{2})} = \tfrac{1}{2}(\Delta_\gamma)^{\frac{D}{2}-2}\Gamma(2-\tfrac{D}{2})\Gamma(\tfrac{D}{2}) \quad .$$

The integral over the solid angle

$$\int \frac{\mathrm{d}\Omega_E^{D-1}}{(2\pi)^D} = \frac{2\pi^{D/2}}{(2\pi)^D\Gamma(\frac{D}{2})} = \frac{2}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}$$

gives then

$$\begin{aligned}
\Pi_\gamma^{[2]}(q^2,D) &= -8e^2\int_0^1 \mathrm{d}x\, x(1-x)\,\frac{2}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\,\tfrac{1}{2}(\Delta_\gamma)^{\frac{D}{2}-2}\Gamma(2-\tfrac{D}{2})\Gamma(\tfrac{D}{2}) \\
&= -\frac{8e^2}{(4\pi)^2}\Gamma(2-\tfrac{D}{2})\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{\Delta_\gamma}{4\pi}\right)^{\frac{D}{2}-2} .
\end{aligned}$$

Using the expansion $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_{\text{E.M.}} + \mathcal{O}(\epsilon)$, where $\gamma_{\text{E.M.}} \sim 0.5772$ is the Euler-Mascheroni constant, with $\epsilon = 4 - D$ we get

$$\begin{aligned}
\Pi_\gamma^{[2]}(q^2,\epsilon) &= -\frac{8e^2}{(4\pi)^2}\Gamma(\tfrac{\epsilon}{2})\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{\Delta_\gamma}{4\pi}\right)^{-\frac{\epsilon}{2}} \\
&= -\frac{8e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma_{\text{E.M.}} + \mathcal{O}(\epsilon)\right]\int_0^1 \mathrm{d}x\, x(1-x)\, e^{-\frac{\epsilon}{2}\ln\left(\frac{\Delta_\gamma}{4\pi}\right)} \\
&= -\frac{8e^2}{(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma_{\text{E.M.}} + \mathcal{O}(\epsilon)\right]\int_0^1 \mathrm{d}x\, x(1-x)\left[1 - \frac{\epsilon}{2}\ln\left(\frac{\Delta_\gamma}{4\pi}\right) + \mathcal{O}(\epsilon^2)\right] \\
&= -\frac{8e^2}{(4\pi)^2}\int_0^1 \mathrm{d}x\, x(1-x)\left[\frac{2}{\epsilon} - \gamma_{\text{E.M.}} + \ln(4\pi) - \ln\Delta_\gamma + \mathcal{O}(\epsilon)\right] .
\end{aligned}$$

Regularisation → Renormalisation

- throwing away $\frac{2}{\epsilon}$ $\Rightarrow$ minimal subtraction MS
- throwing away $\frac{2}{\epsilon} - \gamma_{\text{E.M.}} + \ln(4\pi)$ $\Rightarrow$ modified minimal subtraction $\overline{\text{MS}}$

From there enters the dependence of calculations on the renormalization scheme.

Λ_{QCD} **is scheme dependent!** A change from scheme A to scheme B goes with a change in the strong coupling:

$$\alpha_s^B = \alpha_s^A(1 + c_1\alpha_s^A + \dots) ,$$

so

$$\begin{aligned}
\ln\frac{\Lambda^B}{\Lambda^A} &= \tfrac{1}{2}\int_{\alpha_s^A(|q^2|)}^{\alpha_s^B(|q^2|)} \frac{\mathrm{d}\alpha}{b\alpha^2(1+\dots)} = \frac{1}{2b}\left[-\frac{1}{\alpha}(1+\dots)\right]_{\alpha_s^A(|q^2|)}^{\alpha_s^B(|q^2|)} = \frac{1}{2b}\left[\frac{1}{\alpha_s^A} - \frac{1}{\alpha_s^B}\right](1+\dots) \\
&= \frac{1}{2b}\left[\frac{\alpha_s^B - \alpha_s^A}{\alpha_s^A\alpha_s^B}\right](1+\dots) = \frac{1}{2b}\left[\frac{\alpha_s^A + c_1(\alpha_s^A)^2 + \dots - \alpha_s^A}{(\alpha_s^A)^2(1 + c_1\alpha_s^A + \dots)}\right](1+\dots) \to \frac{c_1}{2b} ,
\end{aligned}$$

when $|q^2| \to \infty$, as $\alpha_s \to 0$ and $\ln\frac{\Lambda^B}{\Lambda^A}$ becomes independent of $|q^2|$. With this we have

$$\Lambda^2_{\overline{\text{MS}}} = \Lambda^2_{\text{MS}} e^{\ln\frac{4}{\pi} - \gamma_{\text{E.M.}}}$$

$\Rightarrow$ better define Λ_{QCD} at M_Z^2

$\sigma(e^+e^- \to \text{hadrons})$ **revisited**

We had the perturbation series in α_s for cross section

$$\sigma = \sigma_{\text{point}}\left[1 + \frac{\alpha_s(\mu^2)}{\pi} + \sum_{n=2}^{\infty} C_n(\tfrac{s}{\mu^2})\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n\right]$$

with values calculated $C_2(1) = 1.411$ and $C_3(1) = -12.8$. We can fix the μ^2 dependence for each coefficient by $\frac{\partial\sigma}{\partial\mu^2} = 0$. When we truncate after $n = 2$ we have

$$\sigma = \sigma_{\text{point}}\left[1 + \frac{\alpha_s(\mu^2)}{\pi} + C_2(\tfrac{s}{\mu^2})\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\right] .$$

Applying the logarithmic derivative

$$\mu^2 \frac{\mathrm{d}(\sigma/\sigma_{\mathrm{point}})}{\mathrm{d}\mu^2} = 0 = \frac{1}{\pi}\mu^2 \frac{\mathrm{d}\alpha_s(\mu^2)}{\mathrm{d}\mu^2} + \mu^2 \frac{\mathrm{d}C_2(\frac{s}{\mu^2})}{\mathrm{d}\mu^2} \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 + C_2(\tfrac{s}{\mu^2}) 2 \frac{\alpha_s(\mu^2)}{\pi} \mu^2 \frac{\mathrm{d}\alpha_s(\mu^2)}{\mathrm{d}\mu^2}$$

or

$$\mu^2 \frac{\mathrm{d}C_2(\frac{s}{\mu^2})}{\mathrm{d}\mu^2} = -\frac{\pi^2}{\alpha_s^2}\left[\frac{1}{\pi}\beta(\alpha_s) + C_2(\tfrac{s}{\mu^2}) 2 \frac{\alpha_s}{\pi}\beta(\alpha_s)\right] = -\frac{\pi\beta(\alpha_s)}{\alpha_s^2} = -\frac{\pi(-b\alpha_s^2)}{\alpha_s^2} = \pi b \ ,$$

if we ignore the second term, as it is of a higher order, and insert the value of β to one-loop order. Setting the integration variable $x = \frac{s}{\mu^2}$ we get $\mathrm{d}x = -\frac{s\mathrm{d}\mu^2}{(\mu^2)^2}$ or $\frac{\mathrm{d}\mu^2}{\mu^2} = -\frac{\mathrm{d}x}{x}$ and

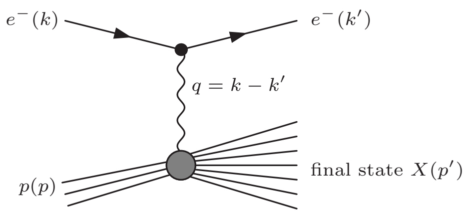
$$\int_1^{\frac{s}{\mu^2}} \mathrm{d}C_2(x) = C_2(\tfrac{s}{\mu^2}) - C_2(1) = \int_1^{\frac{s}{\mu^2}} \frac{\mathrm{d}\mu^2}{\mu^2}\pi b = \pi b \int_1^{\frac{s}{\mu^2}} (-\frac{\mathrm{d}x}{x}) = -\pi b \ln \tfrac{s}{\mu^2} \ ,$$

and hence the energy dependent shift of the coefficient

$$C_2(\tfrac{s}{\mu^2}) = C_2(1) - \pi b \ln \tfrac{s}{\mu^2} \ .$$

QCD corrections to the Parton Model

As a reminder: the Bjorken $x = \frac{Q^2}{2M\nu}$ describes the scaling of the structure functions: $MW_1(Q^2, \nu) \to F_1(x)$, which has no seperate dependence on the variables Q^2 and ν.

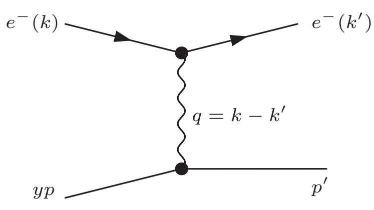


This diagram illustrates the definition of the structure function

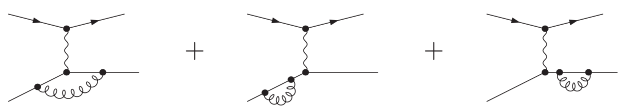
$$\begin{aligned} e^2 W^{\mu\nu}(q,p) &= \frac{1}{4\pi M} \tfrac{1}{2} \sum_s \sum_{X,p'} \langle p(p,s)|\hat{j}^\mu_{\mathrm{em}}(0)|X(p')\rangle\langle X(p')|\hat{j}^\nu_{\mathrm{em}}(0)|p(p,s)\rangle \times (2\pi)^4 \delta^4(p+q-p') \\ &= e^2\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1(Q^2,\nu) + e^2\left[p^\mu - \frac{(p.q)q^\mu}{q^2}\right]\left[p^\nu - \frac{(p.q)q^\nu}{q^2}\right]\frac{1}{M^2} W_2(Q^2,\nu) \end{aligned}$$

with $p.q) = M\nu$ and $q^2 = -Q^2$.

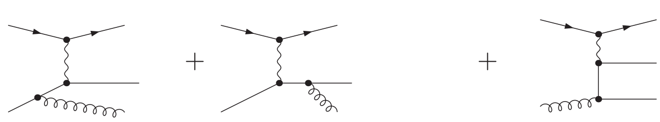
On the parton level the diagram simplifies to



where yp is the momentum fraction of the parton. It has virtual



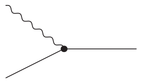
and real corrections



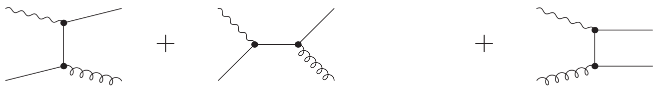
…but this last contribution can be subtracted by studying the difference $(ep) - (en)$ of the electron scattering between proton and neutron $\longrightarrow$ "non-singlet" contributions.

Everything is done on the cross section level: i.e.: it is summed over the final spins and averaged over the initial spins. That means, that no information about the electron is retained. Then it is simpler to study the virtual photon scattering – but with only transverse photons!

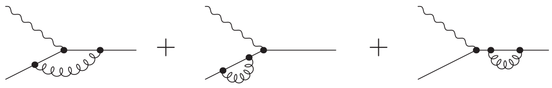
The real diagrams



and



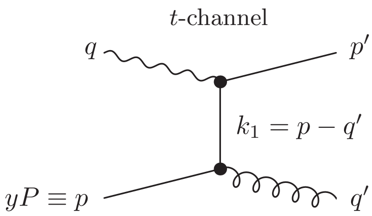
and the virtual diagrams



to calculate the higher orders of the structure function. Writing the phase space factor $K = \frac{W^2-M^2}{2M}$, where W^2 is the invariant mass of the hadronic final state, we can relate eq.(9.43) and eq.(9.45)

$$W_1 = \frac{\sigma_T}{4\pi\alpha^2/K} \qquad \text{with} \quad \sigma_T = \frac{1}{2}\sum_{\lambda=\pm 1} \varepsilon^*_\mu(\lambda)\varepsilon_\nu(\lambda)W^{\mu\nu}(q,p)$$

Calculating $\gamma q \to gq$: in the CM-frame, as it is the real correction to W_1. The matrix element comes from the diagrams



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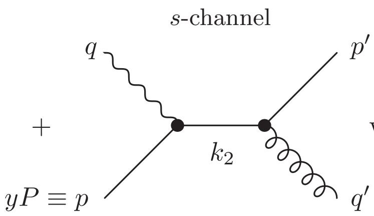


with $k_2 = p' + q'$

and reads

$$\begin{aligned}
\mathcal{M}_{\gamma q\to gq} &= -e_q g_s \varepsilon^*_\nu(q',\lambda')\varepsilon_\mu(q,\lambda)\bar{u}(p',s')\left[\gamma^\mu \frac{i}{\not{k}_1 - m_q}\gamma^\nu + \gamma^\nu \frac{i}{\not{k}_2 - m_q}\gamma^\mu\right] u(p,s) \\
&= -ie_q g_s \varepsilon^*_\nu(q',\lambda')\varepsilon_\mu(q,\lambda)\bar{u}(p',s')\left[\frac{F_t^{\mu\nu}}{t} + \frac{F_s^{\mu\nu}}{s}\right] u(p,s)
\end{aligned}$$

where we use the Mandelstam variables $s = (p'+q')^2$ and $t = (p'-q)^2$ and the small quark masses to reduce the squares of the propagators to $\frac{1}{s}$ and $\frac{1}{t}$, respectively. The spin averaged matrix element squared $\langle|\mathcal{M}_{\gamma q\to gq}|^2\rangle$ is then

$$\begin{aligned}
\frac{1}{4}\sum_{s,s',\lambda,\lambda'} |\mathcal{M}_{\gamma q\to gq}|^2 &= \frac{e_q^2 g_s^2}{4}\sum_{s,s',\lambda,\lambda'} (\varepsilon^*_\nu(q',\lambda')\varepsilon_{\nu'}(q',\lambda'))(\varepsilon_\mu(q,\lambda)\varepsilon^*_{\mu'}(q,\lambda)) \\
&\quad \times \bar{u}(p',s')\left[\frac{F_t^{\mu\nu}}{t} + \frac{F_s^{\mu\nu}}{s}\right] u(p,s)\left(\bar{u}(p',s')\left[\frac{F_t^{\mu'\nu'}}{t} + \frac{F_s^{\mu'\nu'}}{s}\right] u(p,s)\right)^* .
\end{aligned}$$

The spin sum for the transverse photon is the same as for the real photon or the real gluon:

$$\sum_{\lambda'}(\varepsilon^*_\nu(q',\lambda')\varepsilon_{\nu'}(q',\lambda')) = (-g_{\nu\nu'}) \qquad \text{and} \qquad \sum_{\lambda}(\varepsilon_\mu(q,\lambda)\varepsilon^*_{\mu'}(q,\lambda)) = (-g_{\mu\mu'}) \ .$$

The complex conjugate amplitude can be rewritten as

$$\begin{aligned}
&\left(\bar{u}(p')\left[\tfrac{F_t^{\mu'\nu'}}{t} + \tfrac{F_s^{\mu'\nu'}}{s}\right] u(p)\right)^* = \left(\bar{u}(p')\left[\tfrac{F_t^{\mu'\nu'}}{t} + \tfrac{F_s^{\mu'\nu'}}{s}\right] u(p)\right)^\dagger = u^\dagger(p)\left[\tfrac{(F_t^{\mu'\nu'})^\dagger}{t} + \tfrac{(F_s^{\mu'\nu'})^\dagger}{s}\right]\bar{u}^\dagger(p') \\
= \ & \bar{u}(p)\gamma^0\left[\tfrac{(F_t^{\mu'\nu'})^\dagger}{t} + \tfrac{(F_s^{\mu'\nu'})^\dagger}{s}\right]\gamma^0 u(p') = \bar{u}(p)\left[\tfrac{1}{t}\bar{F}_t^{\mu'\nu'} + \tfrac{1}{s}\bar{F}_s^{\mu'\nu'}\right] u(p') \ ,
\end{aligned}$$

which allows the splitting of the fermion trace

$$\begin{aligned}
&\sum_{s,s'} \bar u(p',s')\left[\tfrac{1}{t}F_t^{\mu\nu}+\tfrac{1}{s}F_s^{\mu\nu}\right]u(p,s)\left(\bar u(p',s')\left[\tfrac{1}{t}F_t^{\mu'\nu'}+\tfrac{1}{s}F_s^{\mu'\nu'}\right]u(p,s)\right)^*\\
=\;&\mathrm{Tr}\left[\sum_{s,s'}\left[\tfrac{1}{t}F_t^{\mu\nu}+\tfrac{1}{s}F_s^{\mu\nu}\right]u(p,s)\bar u(p,s)\left[\tfrac{1}{t}\bar F_t^{\mu'\nu'}+\tfrac{1}{s}\bar F_s^{\mu'\nu'}\right]u(p',s')\bar u(p',s')\right]\\
=\;&\mathrm{Tr}\left[[\tfrac{1}{t}F_t^{\mu\nu}+\tfrac{1}{s}F_s^{\mu\nu}](\not p+m_q)[\tfrac{1}{t}\bar F_t^{\mu'\nu'}+\tfrac{1}{s}\bar F_s^{\mu'\nu'}](\not p'+m_q)\right]\\
=\;&\tfrac{1}{s^2}\mathrm{Tr}\left[F_s^{\mu\nu}\not p\bar F_s^{\mu'\nu'}\not p'\right]+\tfrac{1}{st}\mathrm{Tr}\left[F_s^{\mu\nu}\not p\bar F_t^{\mu'\nu'}\not p'+F_t^{\mu\nu}\not p\bar F_s^{\mu'\nu'}\not p'\right]+\tfrac{1}{t^2}\mathrm{Tr}\left[F_t^{\mu\nu}\not p\bar F_t^{\mu'\nu'}\not p'\right]
\end{aligned}$$

into three parts; we used that the quark masses can be neglected in the last line. Using

$$\begin{aligned}
\tfrac{1}{s}F_s^{\mu\nu} &= \gamma^\nu\frac{1}{\not k_2}\gamma^\mu=\gamma^\nu\frac{(\not p'+\not q')}{(p'+q')^2}\gamma^\mu=\tfrac{1}{s}\gamma^\nu(\not p'+\not q')\gamma^\mu \qquad \text{and}\\
\tfrac{1}{t}F_t^{\mu\nu} &= \gamma^\mu\frac{1}{\not k_1}\gamma^\nu=\gamma^\mu\frac{(\not p-\not q')}{(p-q')^2}\gamma^\nu=\tfrac{1}{t}\gamma^\mu(\not p-\not q')\gamma^\nu
\end{aligned}$$

we get

$$\begin{aligned}
\bar F_s^{\mu\nu} &= \gamma^0(\gamma^\nu(\not p'+\not q')\gamma^\mu)^\dagger\gamma^0=\gamma^0(\gamma^\mu)^\dagger\gamma^0\gamma^0(\not p'+\not q')^\dagger\gamma^0\gamma^0(\gamma^\nu)^\dagger\gamma^0=\gamma^\mu(\not p'+\not q')\gamma^\nu \qquad \text{and}\\
\bar F_t^{\mu\nu} &= \gamma^0(\gamma^\mu(\not p-\not q')\gamma^\nu)^\dagger\gamma^0=\gamma^0(\gamma^\nu)^\dagger\gamma^0\gamma^0(\not p-\not q')^\dagger\gamma^0\gamma^0(\gamma^\mu)^\dagger\gamma^0=\gamma^\nu(\not p-\not q')\gamma^\mu\ .
\end{aligned}$$

Contracting with the spin sums of photon ($-g_{\mu\mu'}$) and gluon ($-g_{\nu\nu'}$) gives for the traces

$$\begin{aligned}
&\mathrm{Tr}\left[F_s^{\mu\nu}\not p\bar F^s_{\mu\nu}\not p'\right]=\mathrm{Tr}\left[\gamma^\nu(\not p'+\not q')\gamma^\mu\not p\gamma_\mu(\not p'+\not q')\gamma_\nu\not p'\right]=\mathrm{Tr}\left[(\not p'+\not q')(-2\not p)(\not p'+\not q')\gamma_\nu\not p'\gamma^\nu\right]\\
&\quad=\;4\mathrm{Tr}\left[(\not p'+\not q')\not p(\not p'+\not q')\not p'\right]=4\mathrm{Tr}\left[\not q'\not p\not q'\not p'\right]=4\mathrm{Tr}\left[\not q'(2(p.q')-\not q'\not p)\not p'\right]=8(p.q')\mathrm{Tr}\left[\not q'\not p'\right]\\
&\quad=\;4(-t)4(p'.q')=-8st\ ,
\end{aligned}$$

since $2(p.q')=-((p-q')^2-p^2-q'^2)=-t$ and $2(p'.q')=((p'+q')^2-p'^2-q'^2)=s$.

$$\begin{aligned}
&\mathrm{Tr}\left[F_t^{\mu\nu}\not p\bar F^t_{\mu\nu}\not p'\right]=\mathrm{Tr}\left[\gamma^\mu(\not p-\not q')\gamma^\nu\not p\gamma_\nu(\not p-\not q')\gamma_\mu\not p'\right]=\mathrm{Tr}\left[(\not p-\not q')(-2\not p)(\not p-\not q')(-2\not p')\right]\\
&\quad=\;4\mathrm{Tr}\left[\not q'\not p\not q'\not p'\right]=-8st\ ,
\end{aligned}$$

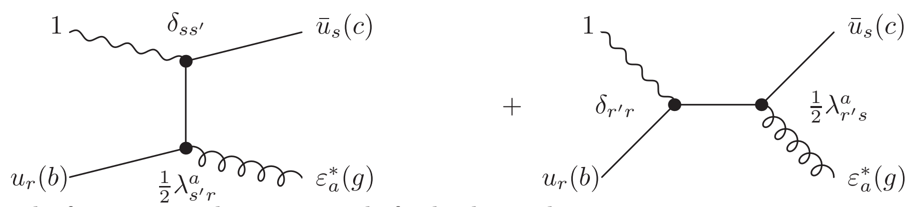
and

$$\begin{aligned}
&\mathrm{Tr}\left[F_s^{\mu\nu}\not p\bar F^t_{\mu\nu}\not p'+F_t^{\mu\nu}\not p\bar F^s_{\mu\nu}\not p'\right]=\mathrm{Tr}\left[\gamma^\nu(\not p'+\not q')\gamma^\mu\not p\gamma_\nu(\not p-\not q')\gamma_\mu\not p'+\gamma^\mu(\not p-\not q')\gamma^\nu\not p\gamma_\mu(\not p'+\not q')\gamma_\nu\not p'\right]\\
&\quad=\;\mathrm{Tr}\left[(-2\not p\gamma^\mu(\not p'+\not q'))(\not p-\not q')\gamma_\mu\not p'+\gamma^\mu(\not p'-\not q)(-2(\not p'+\not q')\gamma_\mu\not p)\not p'\right]\\
&\quad=\;-2\mathrm{Tr}\left[(\not p'+\not q')(\not p-\not q')\gamma_\mu\not p'\not p\gamma^\mu+(\not p-\not q')(\not p'+\not q')\gamma_\mu\not p\not p'\gamma^\mu\right]\\
&\quad=\;-2\mathrm{Tr}\left[(\not p'+\not q')(\not p-\not q')4(p'.p)+(\not p-\not q')(\not p'+\not q')4(p.p')\right]\\
&\quad=\;4u\mathrm{Tr}\left[\not p'\not p+\not q'\not p-\not p'\not q'-\not q'\not q'+\not p\not p'+\not p\not q'-\not q'\not p'-\not q'\not q'\right]=32u((p'.p)+(q'.p)-(p'.q'))\\
&\quad=\;16u((-u)+(-t)-s)=-16u(s+t+u)=-16uq^2\ ,
\end{aligned}$$

since all the other momentum squares are zero. So

$$\langle|\mathcal{M}_{\gamma q\to gq}|^2\rangle \;=\; \tfrac{e_q^2g_s^2}{4}(\tfrac{1}{s^2}(-8st)+\tfrac{1}{t^2}(-8st)+\tfrac{1}{st}(16uq^2)=-2e_q^2g_s^2(\tfrac{t}{s}+\tfrac{s}{t}+2q^2\tfrac{u}{st})\ ,$$

which is the same result as in Compton scattering. But we are still missing the color factor:



These colorfactors give the same result for both graphs:

$$\varepsilon^*_a(g)\bar u_s(c)\delta_{ss'}\tfrac12\lambda^a_{s'r}u_r(b)=\tfrac12\varepsilon^*_a(g)(\bar u(c)\lambda^a u(b))=\varepsilon^*_a(g)\bar u_s(c)\tfrac12\lambda^a_{sr'}\delta_{r'r}u_r(b)$$

Writing the fundamental representation $u_s(b)$ with three basis vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} ,$$

its hermitian cobjugate with the corresponding row vectors and the adjoint representation $\varepsilon_a(g)$ with 8-dimensional unit vectors, we can express the sum over colors

$$\sum_{c=r,b,g} u_s(c)\bar{u}_{s'}(c) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot (1,0,0) + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot (0,1,0) + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot (0,0,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ss'}$$

and similarly

$$\sum_{g=1}^{8} \varepsilon_a(g)\varepsilon^*_{a'}(g) = \delta_{aa'} \ ,$$

so that the average over the incoming and the sum over the outgoing colors becomes

$$\begin{aligned} \frac{1}{3}\sum_{b,c,g} |\tfrac{1}{2}\varepsilon^*_a(g)(\bar{u}(c)\lambda^a u(b))|^2 &= \frac{1}{12}\sum_{b,c,g} \varepsilon^*_a(g)\bar{u}_r(c)\lambda^a_{rs}u_s(b)\varepsilon_{a'}(g)\bar{u}_{r'}(b)[(\lambda^{a'})^\dagger]_{r's'}u_{s'}(c)) \\ &= \tfrac{1}{12}\delta_{aa'}\delta_{sr'}\delta_{rs'}\lambda^a_{rs}\lambda^{a'}_{r's'} = \tfrac{1}{12}\lambda^a_{rs}\lambda^a_{sr} = \tfrac{1}{12}\mathrm{Tr}[\lambda^a\lambda^a] = \tfrac{1}{12}16 = \tfrac{4}{3} \ . \end{aligned}$$

Now we need only d$Lips$ to calculate our cross sections:

$$\begin{aligned} \mathrm{d}Lips(p',q') &= (2\pi)^4\delta^4(q+yp-q'-p')\frac{\mathrm{d}^3\vec{p}'}{(2\pi)^3 2E_{p'}}\frac{\mathrm{d}^3\vec{q}'}{(2\pi)^3 2E_{q'}} = \frac{\delta(q^0+yp^0-E_{q'}-E_{p'})\mathrm{d}^3\vec{q}'}{(2\pi)^2 2E_{p'}2E_{q'}} \\ &= \frac{\delta(q^0+yp^0-E_{q'}-E_{p'})}{(4\pi)^2 E_{p'}E_{q'}} \ |\vec{q}'|^2\mathrm{d}|\vec{q}'| \ \mathrm{d}^2\Omega_{q'} \end{aligned}$$

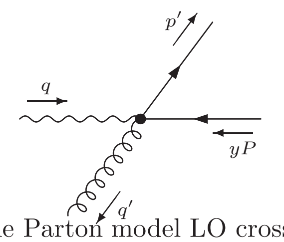
From $q'^2=0$ follows $|\vec{q}'| = E_{q'} =: E$ and $\mathrm{d}|\vec{q}'| = \mathrm{d}E$ and from $p'^2=0$ follows $E_{p'}| = \vec{p}'| = |\vec{q}'| = E$ and

$$\mathrm{d}Lips(p',q') = \frac{\delta(q^0+yp^0-2E)}{(4\pi)^2E^2} \ E^2\mathrm{d}E \ \mathrm{d}^2\Omega_{q'} = \frac{1}{(4\pi)^2}\frac{1}{2}\delta(\frac{q^0+yp^0}{2}-E) \ \mathrm{d}E\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi \ .$$

So

$$\begin{aligned} e^2W_1 &= \int \tfrac{1}{2}\tfrac{4}{3}[-2e_q^2g_s^2(\tfrac{t}{s}+\tfrac{s}{t}-2q^2\tfrac{u}{st})]\tfrac{1}{(4\pi)^2}\tfrac{1}{2}\delta(\tfrac{q^0+yp^0}{2}-E) \ \mathrm{d}E\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi \\ &= -\frac{2\pi}{3}\frac{2e_q^2g_s^2}{(4\pi)^2}\int \mathrm{d}\cos\theta\,(\frac{t}{s}+\frac{s}{t}+2q^2\frac{u}{st}) = -e_q^2\frac{\alpha_s}{3}\int \mathrm{d}\cos\theta\,(\frac{t}{s}+\frac{s}{t}-2Q^2\frac{u}{st}) \ , \end{aligned}$$

where the 2π come from the integral over dφ and the energy E is fixed to $\frac{1}{2}(q^0+yp^0)$. The integral over $\mathrm{d}\cos\theta$ has to be done in the CM-frame of the NLO correction



From the Parton model LO cross section we calculate the correction to the process $\gamma + q \to q'$, where the quark q has the momentum fraction yP. The correction is the gluon radiation, so that the quark has only a fraction of this original momentum fraction yP. We will call this reduced fraction yzP and hence we

$$q + yzP = p' \ .$$

The quark should stay on-shell, so we have $P^2 = p'^2 = 0$ and the energy transfer from the photon was $q^2 = Q^2$. So we get

$$0 = (q+yzP)^2 = q^2 + 2yz(P.q) + y^2z^2P^2 = -Q^2 + 2yz(P.q) \qquad \text{or} \quad 2y(P.q) = \frac{Q^2}{z} \ .$$

For the correction we have the process $\gamma + q \to g + q'$ and the four-momentum conservation

$$q + yP = p' + q' \ .$$

This gives the Mandelstam variables of this process

$$\begin{aligned} s &= (p'+q')^2 = (q+yP)^2 = q^2 + 2y(P.q) + y^2P^2 = -Q^2 + \frac{Q^2}{z} = \frac{Q^2}{z}(1-z) \\ t &= (q-p')^2 = (q'-yP)^2 = -2y(P.q') \\ u &= (q-q')^2 = (p'-yP)^2 = -2y(P.p') \end{aligned}$$

In the CM-frame we assigne the momenta

$$\begin{aligned} q &= (q^0,0,0,k) & p' &= k'(1,\sin\theta,0,\cos\theta) \\ yP &= k(1,0,0,-1) & q' &= k'(1,-\sin\theta,0,-\cos\theta) \end{aligned}$$

and get for the Mandelstam variables

$$\begin{aligned} s &= (p'+q')^2 = (2k')^2 = 4k'^2 = \frac{Q^2}{z}(1-z) \\ t &= -2(yP.q') = -2kk'(1-\cos\theta) \\ u &= -2(yP.p') = -2kk'(1+\cos\theta) \end{aligned}$$

With the energy conservation $q^0 + k = 2k'$ we get $q^0 = 2k' - k$ and

$$q^2 = -Q^2 = (q^0)^2 - k^2 = (2k'-k)^2 - k^2 = 4k'^2 - 4kk' + k^2 - k^2 = s - 4kk' = \frac{Q^2}{z}(1-z) - 4kk'$$

so $0 = \frac{Q^2}{z} - 4kk'$ or $2kk' = \frac{Q^2}{2z}$ and

$$s = \frac{Q^2}{z}(1-z) \qquad t = -\frac{Q^2}{2z}(1-\cos\theta) \qquad u = -\frac{Q^2}{2z}(1+\cos\theta)$$

and so

$$\begin{aligned} e^2W_1 &= -e_q^2\frac{\alpha_s}{3}\int_{-1}^{1}\mathrm{d}\cos\theta\,(\frac{-\frac{Q^2}{2z}(1-\cos\theta)}{\frac{Q^2}{z}(1-z)} + \frac{\frac{Q^2}{z}(1-z)}{-\frac{Q^2}{2z}(1-\cos\theta)} - 2Q^2\frac{\frac{Q^2}{2z}(1+\cos\theta)}{\frac{Q^2}{z}(1-z)\frac{Q^2}{2z}(1-\cos\theta)}) \\ &= -e_q^2\frac{\alpha_s}{3}\int_{-1}^{1}\mathrm{d}c\left(\frac{1}{2}\frac{1-c}{1-z} + 2\frac{1-z}{1-c} - 2\frac{z}{1-z}\frac{1+c}{1-c}\right) \quad \to \infty \quad \text{for the upper boundary} \ . \end{aligned}$$

$\Rightarrow$ collinear mass divergence — but only in the zero quark mass limit!

When we include the quark mass in the propagator:

$$\frac{1}{t} = \frac{1}{(q-p')^2} \to \frac{1}{(q-p')^2 - m_q^2} = \frac{1}{t - m_q^2} = \frac{1}{-\frac{Q^2}{2z}(1-c) - m_q^2} = \frac{-\frac{2z}{Q^2}}{1 + \frac{2m_q^2z}{Q^2} - c} \ ,$$

which we can now integrate (with $a = 1 + \frac{2m_q^2z}{Q^2}$):

$$\begin{aligned} -\frac{2z}{Q^2}\int_{-1}^{1}\frac{\mathrm{d}c}{a-c} &= -\frac{2z}{Q^2}\cdot[-\ln(a-c)]_{-1}^{1} = \frac{2z}{Q^2}\ln\frac{a-1}{a+1} = \frac{2z}{Q^2}\ln\frac{1+\frac{2m_q^2z}{Q^2}-1}{1+\frac{2m_q^2z}{Q^2}+1} = \frac{2z}{Q^2}\ln\frac{\frac{m_q^2z}{Q^2}}{1+\frac{m_q^2z}{Q^2}} \\ &\sim \frac{2z}{Q^2}\ln\frac{m_q^2z}{Q^2} \qquad \text{for } Q^2 \to \infty \ , \end{aligned}$$

which gives the expected large logarithms.

Keeping the masses in the denominators

$$\frac{1}{t} \to \frac{-\frac{2z}{Q^2}}{a-c} \qquad \text{and} \qquad \frac{1}{s} \to \frac{1}{(p'+q')^2 - m_q^2} = \frac{1}{s - m_q^2}$$

we get for e^2W_1

$$
\begin{aligned}
&= -e_q^2\frac{\alpha_s}{3}\int_{-1}^{1}\mathrm{d}c\,(\frac{1}{t}[s-2Q^2\frac{u}{s}]+\frac{t}{s}) \quad\rightarrow\quad -e_q^2\frac{\alpha_s}{3}\int_{-1}^{1}\mathrm{d}c\,(\frac{-\frac{2z}{Q^2}}{a-c}[s-\frac{2Q^2u}{s-m_q^2}]+\frac{t}{s-m_q^2}) \\
&= e_q^2\frac{\alpha_s}{3}\int_{-1}^{1}\mathrm{d}c\left(\frac{\frac{2z}{Q^2}}{a-c}[\frac{Q^2}{z}(1-z)-\frac{2Q^2(-\frac{Q^2}{2z}(1+c))}{\frac{Q^2}{z}(1-z)-m_q^2}]-\frac{-\frac{Q^2}{2z}(1-c)}{\frac{Q^2}{z}(1-z)-m_q^2}\right) \\
&= e_q^2\frac{\alpha_s}{3}\int_{-1}^{1}\mathrm{d}c\left(\frac{2}{a-c}\left[(1-z)+\frac{(1+c)z}{(1-z)-\frac{zm_q^2}{Q^2}}\right]+\frac{1}{2}\frac{(1-c)}{(1-z)-\frac{zm_q^2}{Q^2}}\right) \\
&= e_q^2\frac{\alpha_s}{3}\int_{-1}^{1}\mathrm{d}c\left(\frac{2}{a-c}\frac{(1-z)[(1-z)-\frac{zm_q^2}{Q^2}]+(1+a-(a-c))z}{(1-z)-\frac{zm_q^2}{Q^2}}+\frac{1}{2}\frac{(1-c)}{(1-z)-\frac{zm_q^2}{Q^2}}\right) \\
&= e_q^2\frac{\alpha_s}{3}\int_{-1}^{1}\mathrm{d}c\left(\frac{2}{a-c}\frac{(1-z)[1-z-\frac{zm_q^2}{Q^2}]+(1+a)z}{(1-z)-\frac{zm_q^2}{Q^2}}+(\frac{1}{2}-2z)\frac{1}{1-z-\frac{zm_q^2}{Q^2}}-\frac{c}{2}\frac{1}{1-z-\frac{zm_q^2}{Q^2}}\right) \\
&= e_q^2\frac{\alpha_s}{3}2\frac{(1-z)[1-z-\frac{zm_q^2}{Q^2}]+(1+1+\frac{2m_q^2z}{Q^2})z}{(1-z)-\frac{zm_q^2}{Q^2}}\left(\int_{-1}^{1}\frac{\mathrm{d}c}{a-c}\right)+e_q^2\frac{\alpha_s}{3}(\frac{1}{2}-2z)\frac{1}{1-z-\frac{zm_q^2}{Q^2}}\left(\int_{-1}^{1}\mathrm{d}c\right) \\
&\quad -\frac{1}{1-z-\frac{zm_q^2}{Q^2}}\left(\int_{-1}^{1}\mathrm{d}c\,\frac{c}{2}\right) \\
&= e_q^2\frac{2\alpha_s}{3}\frac{(1-z)^2+2z-(1-2z)\frac{zm_q^2}{Q^2}}{(1-z)-\frac{zm_q^2}{Q^2}}\ln\frac{1+\frac{m_q^2z}{Q^2}}{\frac{m_q^2z}{Q^2}}+e_q^2\frac{\alpha_s}{3}\frac{1-4z}{1-z-\frac{zm_q^2}{Q^2}} \\
&\quad \xrightarrow{Q^2\gg m_q^2} \qquad e_q^2\alpha_s\frac{2}{3}\frac{1+z^2}{1-z}\ln\frac{Q^2}{m_q^2z}\quad .
\end{aligned}
$$

We have to add this correction now to the proton structure function $2F_i = \frac{1}{x}F_2$, which was obtained in chapter 9:

$$\frac{F_2}{x} = 2F_i = \sum_i\int_0^1 \mathrm{d}y f_i(y)\int_0^1 \mathrm{d}z\, 2F_1^i\delta(x-yz)$$

with a reminder from eq.(9.31):

$$W_2(\nu,Q^2) = \sum_i\int_0^1 \mathrm{d}x f_i(x)e_i^2\delta(\nu-\frac{Q^2}{2Mx}) = \sum_i\int_0^1 \mathrm{d}x f_i(x)e_i^2\frac{x}{\nu}\delta(x-\frac{Q^2}{2M\nu})$$

So the correction becomes (when retaining only terms $\sim\frac{1}{t}$, as calculated):

$$\sim\frac{e_q^2\alpha_s}{2\pi}\cdot\frac{4}{3}\frac{1+z^2}{1-z}\cdot\ln\frac{Q^2}{m_q^2} = \frac{e_q^2\alpha_s}{2\pi}\cdot P_{qq}(z)\cdot\ln\frac{Q^2}{m_q^2}\ ,$$

where $P_{qq}(z)$ denotes the **splitting function**, i.e. the probability for a quark to radiate off a gluon while scattering on the virtual photon.

So analogously to summing over all quarks and all contributing momenta, we have also to sum over the energies, that are lost to gluon radiation. Since we **do not know** the quark distribution function, we just add the "mass singularity", i.e. the large logarithm $\ln\frac{Q^2}{m_q^2}$, to the unknown function. Then we have only known and finite terms left and a function, that has to be taken fromthe experiment anyway.

Wit this ansatz we have a quark distribution function

$$q(x,\mu^2) = q(x)+\frac{\alpha_s}{2\pi}\int_x^1\frac{\mathrm{d}y}{y}q(y)\left[P_{qq}(\tfrac{x}{y})\cdot\ln\frac{Q^2}{m_q^2}+C(\tfrac{x}{y})\right]$$

depending on a renormalisation scale μ^2 and a finite function $C(\frac{x}{y})$, which is calculable and depends on the adopted scheme, how to split $\ln\frac{Q^2}{m_q^2}$ into $\ln\frac{Q^2}{\mu^2}+\ln\frac{\mu^2}{m_q^2}$ to make both logarithms calculable. But now this μ^2 is the "factorization scale"! $\Rightarrow$ it is **convenient** to identify the factorization scale with the renormalization scale — but it is **not necessary!**

There are different factorization schemes, to decide which finite parts should be calculated and which other finite parts should be put into the uncalculable, but measureable parton distribution function:

- DIS the deep inelastic scattering scheme: put everything into $q(x,\mu^2)$, then one has
$$2F_1^i(x,Q^2) = e_i^2 q(x,Q^2)$$
- $\overline{\mathrm{MS}}$ the modified minimal subtraction scheme, which adopts the renormalization prescription. Then one has
$$F_1(x,Q^2) = e_i^2 \int_x^1 \frac{\mathrm{d}y}{y} q(y,Q^2)\left[\delta(1-\tfrac{x}{y}) + \frac{\alpha_s}{2\pi} C_{\overline{\mathrm{MS}}}(\tfrac{x}{y})\right] \ ,$$
where the $\delta(1-\frac{x}{y})$ comes from the LO contribution, i.e. the same as in the DIS scheme.

Repeating the RGE procedure to the parton distribution functions (PDFs), we can write for a single quark
$$\mu^2 \frac{\partial q(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} P_{qq}(\tfrac{x}{y}) q(y,\mu^2) \ .$$
When including all quarks and the gluon, one gets coupled differential equations for the PDFs.

How do we measure the PDFs? ... by measuring different cross sections at different energies.

A convenient way to parametrize the PDFs is by Mellin moments:
$$M_q^{[n]}(t) = \int_0^1 \mathrm{d}x\, x^{n-1} q(x,t) \qquad \text{with} \quad t = \ln \mu^2 \ .$$
Using the RGE
$$\mu^2 \frac{\partial q(x,\mu^2)}{\partial \mu^2} = \frac{\partial}{\partial t} q(x,t) = \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{\mathrm{d}y}{y} P_{qq}(\tfrac{x}{y}) q(y,t) \ .$$
and taking the moments
$$\int_0^1 \mathrm{d}x\, x^{n-1} \frac{\partial}{\partial t} q(x,t) = \frac{\alpha_s(t)}{2\pi} \int_0^1 \mathrm{d}x\, x^{n-1} \int_x^1 \frac{\mathrm{d}y}{y} P_{qq}(\tfrac{x}{y}) q(y,t) \ .$$
and interchanging the integrations
$$\frac{\partial}{\partial t} \int_0^1 \mathrm{d}x\, x^{n-1} q(x,t) = \frac{\alpha_s(t)}{2\pi} \int_0^1 \mathrm{d}y\, y^{n-1} q(y,t) \int_0^y \frac{\mathrm{d}x}{y} \left(\frac{x}{y}\right)^{n-1} P_{qq}(\tfrac{x}{y})$$
and redefining $x = yz$, so $\mathrm{d}x = y\mathrm{d}z$, in the second integral one gets
$$\frac{\mathrm{d}}{\mathrm{d}t} M_q^{[n]}(t) = \frac{\alpha_s(t)}{2\pi} M_q^{[n]}(t) \int_0^1 \mathrm{d}z\, z^{n-1} P_{qq}(z) =: \frac{\alpha_s(t)}{2\pi} M_q^{[n]}(t) \frac{1}{4} \gamma_{qq}^n \ ,$$
where the last integral was defined like an "anomalous dimension". So instead of a convolution of integrals we have just a product. Using the one loop approximation
$$\frac{\mathrm{d}\alpha_s(t)}{\mathrm{d}t} = -b\alpha_s^2(t) \qquad \text{with} \quad b = \frac{33-2N_f}{12\pi}$$
we can rewrite the equation for the moments:
$$\frac{\mathrm{d}M_q^{[n]}(t)}{M_q^{[n]}(t)} = \frac{\gamma_{qq}^n}{8\pi} \cdot \alpha_s(t)\mathrm{d}t = \frac{\gamma_{qq}^n}{8\pi} \frac{\mathrm{d}\alpha_s}{-b\alpha_s} = -\frac{\gamma_{qq}^n}{8\pi b} \mathrm{d}\ln\alpha_s \ ,$$
or
$$\frac{\mathrm{d}\ln M_q^{[n]}(t)}{\mathrm{d}\ln\alpha_s} = -\frac{\gamma_{qq}^n}{8\pi b} =: -d_{qq}^n$$
with the solution
$$M_q^{[n]}(t) = M_q^{[n]}(t_0) \left(\frac{\alpha_s(t_0)}{\alpha_s(t)}\right)^{d_{qq}^n} \ .$$

The technical problems of this approach were solved 1978.

Physical arguments, like "the net number of quarks is conserved" ..., fixes the gluon contribution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 \mathrm{d}x\, q(x,t)\right] = 0 \qquad \Rightarrow \qquad \int_0^1 \mathrm{d}x\, P_{qq}^{+}(x,t) = 0 \ ,$$

where P_{qq}^{+} is a distribution, defined as

$$\int_0^1 \mathrm{d}z\, f(z) P_{qq}^{+}(z) := \int_0^1 \mathrm{d}z\, [f(z) - f(1)] P_{qq}(z) \ .$$

With these we get for the anomalous dimension

$$\gamma_{qq}^{n} = 4\int_0^1 \mathrm{d}z\, z^{n-1} P_{qq}(z) \ ,$$

using the distribution

$$\int_0^1 \mathrm{d}z\, P_{qq}^{+}(z) = 0 \qquad \text{and} \qquad \int_0^1 \mathrm{d}z\, z^{n-1} P_{qq}^{+}(z) = \int_0^1 \mathrm{d}z\, [z^{n-1} - 1] P_{qq}(z)$$

and the one loop result for the splitting function $P_{qq}(z) = \frac{4}{3}\frac{1+z^2}{1-z}$

$$\begin{aligned}
-\frac{3}{16}\gamma_{qq}^{n} &= -\frac{3}{16}\cdot 4\int_0^1 \mathrm{d}z\, [z^{n-1}-1]\frac{4}{3}\frac{1+z^2}{1-z} = \int_0^1 \mathrm{d}z\, \frac{1-z^{n-1}}{1-z}(1+z^2) = \sum_{j=0}^{n-2}\int_0^1 \mathrm{d}z\, z^j(1+z^2) \\
&= \sum_{j=0}^{n-2}\int_0^1 \mathrm{d}z\, z^j + z^{j+2} = \sum_{j=0}^{n-2}\left[\frac{z^{j+1}}{j+1} + \frac{z^{j+3}}{j+3}\right]_0^1 = \sum_{j=0}^{n-2}\frac{1}{j+1} + \sum_{j=0}^{n-2}\frac{1}{j+3} \\
&= \sum_{j=1}^{n-1}\frac{1}{j} + \sum_{j=3}^{n+1}\frac{1}{j} = \left(\frac{1}{1} + \left[\sum_{j=2}^{n}\frac{1}{j}\right] - \frac{1}{n}\right) + \left(-\frac{1}{2} + \left[\sum_{j=2}^{n}\frac{1}{j}\right] + \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n} - \frac{1}{2} + \frac{1}{n+1} + 2\sum_{j=2}^{n}\frac{1}{j} = \frac{1}{2} - \frac{1}{n(n+1)} + 2\sum_{j=2}^{n}\frac{1}{j} \ ,
\end{aligned}$$

and

$$\gamma_{qq}^{n} = -\frac{8}{3}\left[1 - \frac{2}{n(n+1)} + 4\sum_{j=2}^{n}\frac{1}{j}\right] \ .$$