Translation and Rotation Operators

- The momentum operator $\vec{P} = -i\frac{\partial}{\partial\vec{x}} = -i\vec{\partial}$ generates translations:
  - in index notation: $P_k = -i\frac{\partial}{\partial x^k} = -i\partial_k$

$$\begin{aligned} e^{ia^k P_k} f(x) &= e^{a^k\partial_k} f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(a^k\partial_k)^n f(x) \\ &= f(x) + a^k\partial_k f(x) + \frac{1}{2}a^j a^k \partial_j\partial_k f(x) + \ldots \end{aligned}$$

  - the Taylorseries of $f(x+a)$ is

$$f(x+a) = f(x) + a^k\partial_k f(x) + \frac{1}{2}a^j a^k \partial_j\partial_k f(x) + \cdots = e^{i\vec{a}\vec{P}} f(x)$$

$\Rightarrow$ the operator $e^{i\vec{a}\vec{P}}$ moves the function f by the amount $\vec{a}$

- The angular momentum operator $\vec{L} = \vec{X} \times \vec{P}$ generates rotations
  - in index notation: $L_j = \epsilon_{jk\ell}\, x^k P_\ell = -i\epsilon_{jk\ell}\, x^k \partial_\ell$
  - or $L_x = i(z\partial_y - y\partial_z)$, $L_y = i(x\partial_z - z\partial_x)$, $L_z = i(y\partial_x - x\partial_y)$

Translation and Rotation Operators

- The components of $\vec{L}$ do not commute:
  - if you rotate around the $\hat{x}$-axis and then around the $\hat{y}$-axis, you get a different result than rotating first around $\hat{y}$ and then $\hat{x}$.
  - mathematically:

$$\begin{aligned}[L_y, L_x] &= i^2[(x\partial_z - z\partial_x)(z\partial_y - y\partial_z) - (z\partial_y - y\partial_z)(x\partial_z - z\partial_x)] \\ &= i^2[(x\partial_y + xz\partial_z\partial_y - xy\partial_z^2 - z^2\partial_x\partial_y + zy\partial_x\partial_z) \\ &\quad -(zx\partial_y\partial_z - z^2\partial_y\partial_x - yx\partial_z^2 + y\partial_x + yz\partial_z\partial_x)] \\ &= i^2[x\partial_y - y\partial_x] = -iL_z\end{aligned}$$

  - or in index notation: $[L_j, L_k] = i\epsilon_{jk\ell}L_\ell$ $\Rightarrow$ Rotationgroup

- but the square $L^2 = \vec{L}\cdot\vec{L} = L_kL_k$ does commute:

$$\begin{aligned}[L^2, L_j] &= L_k[L_k, L_j] + [L_k, L_j]L_k = L_k i\epsilon_{kj\ell}L_\ell + i\epsilon_{kj\ell}L_\ell L_k \\ &= L_h i\epsilon_{hjm}L_m + i\epsilon_{mjh}L_hL_m = i(\epsilon_{hjm} + \epsilon_{mjh})L_hL_m = 0\end{aligned}$$

$\Rightarrow$ use L^2 and L_z to describe quantum mechanical states (particles)

Eigenstates of the Rotationgroup

- We write an eigenstate of the operators L^2 and L_z as $|\lambda, m\rangle$

$$L^2|\lambda, m\rangle = \lambda|\lambda, m\rangle \qquad \text{and} \qquad L_z|\lambda, m\rangle = m|\lambda, m\rangle$$

  – $|f\rangle$ is called a ket and used to denote a quantum mechanical state.

- We define the ladder operators $L_\pm = L_x \mp iL_y$ with

$$[L^2, L_\pm] = [L^2, L_x] \mp i[L^2, L_y] = 0 \qquad \text{and}$$
$$[L_z, L_\pm] = [L_z, L_x] \mp i[L_z, L_y] = iL_y \mp i(-iL_x) = \pm(L_x \mp iL_y) = \pm L_\pm$$

$\Rightarrow$ $L_\pm|\lambda, m\rangle$ is also an eigenstate of L^2 and L_z :

$$\begin{aligned} L^2(L_\pm|\lambda, m\rangle) &= ([L^2, L_\pm] + L_\pm L^2)|\lambda, m\rangle = 0 + L_\pm L^2|\lambda, m\rangle \\ &= L_\pm \lambda|\lambda, m\rangle = \lambda(L_\pm|\lambda, m\rangle) \end{aligned}$$

and

$$\begin{aligned} L_z(L_\pm|\lambda, m\rangle) &= ([L_z, L_\pm] + L_\pm L_z)|\lambda, m\rangle = (\pm L_\pm + L_\pm L_z)|\lambda, m\rangle \\ &= (\pm L_\pm + L_\pm m)|\lambda, m\rangle = (m \pm 1)(L_\pm|\lambda, m\rangle) \end{aligned}$$

Eigenstates of the Rotationgroup

- $L_\pm$ does not change the eigenvalue λ of the state $|\lambda, m\rangle$
- $L_\pm$ changes the eigenvalue m of the state $|\lambda, m\rangle$

$\Rightarrow$ the states $|\lambda, m+n\rangle$ with $n \in Z$ are related

$\Rightarrow$ for each λ there would be ∞ many states unless there is

  * $a = m_{\text{min}}$ with $L_-|\lambda, a\rangle = 0$ and
  * $b = m_{\text{max}}$ with $L_+|\lambda, b\rangle = 0$

- using

$$\begin{aligned} L_\pm L_\mp &= (L_x \mp iL_y)(L_x \pm iL_y) &= L_x^2 \pm iL_xL_y \mp iL_yL_x + L_y^2 \\ &= (L_x^2 + L_y^2 + L_z^2) - L_z^2 \pm i[L_x, L_y] &= L^2 - L_z^2 \pm i(iL_z) \\ &= L^2 - L_z(L_z \pm 1) \end{aligned}$$

we can relate a and b.

Eigenstates of the Rotationgroup

- relating a and b:
  - $0 = L_+L_-|\lambda, a\rangle = (\lambda - (a^2 + a))|\lambda, a\rangle \quad \Rightarrow \quad \lambda = a^2 + a$
  - $0 = L_-L_+|\lambda, b\rangle = (\lambda - (b^2 - b))|\lambda, b\rangle \quad \Rightarrow \quad \lambda = b^2 - b$

$$a(a+1) = b(b-1) \quad \text{or} \quad a = -b$$

- Applying (L_+) n times on the state $|\lambda, a\rangle$ gives $|\lambda, a + n\rangle$
- for some n we have to reach $|\lambda, b\rangle \quad \Rightarrow \quad a + n = b$
- with $a = -b$ we get $-b + n = b$ or $m_{\text{max}} = b = \frac{n}{2}$
- The rotationgroup allows for half integer eigenstates

$$\Rightarrow \quad \text{Spinors}$$

Constructing a spinor from the known number of eigenstates

- we can write the two eigenstates as $\left|\frac{1}{2},\frac{1}{2}\right\rangle$ and $\left|\frac{1}{2},-\frac{1}{2}\right\rangle$ with

$$\left\langle\tfrac{1}{2},\tfrac{1}{2}\middle|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \left\langle\tfrac{1}{2},-\tfrac{1}{2}\middle|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = 1 \qquad \text{and} \qquad \left\langle\tfrac{1}{2},\tfrac{1}{2}\middle|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = 0$$

  – but this is not a matrix representation.

- for a matrix representation we need two independent vectors

  ⇒ at least a two dimensional representation (i.e. two component vectors)

  – we can choose any two complex orthonormal 2d vectors for $\left|\frac{1}{2},\frac{1}{2}\right\rangle$ and $\left|\frac{1}{2},-\frac{1}{2}\right\rangle$

  – an $SU(2)$ rotation allows us to rotate these vectors to

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle \to s^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle \to s^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  – now we have to find the $SU(2)$ generators S_k that

    * give the correct eigenvalues for s^+ and s^-:

$$S_3 s^+ = \tfrac{1}{2} s^+ \qquad \text{and} \qquad S_3 s^- = -\tfrac{1}{2} s^-$$

    * raise s^- to $S^+ s^- = s^+$ and lower s^+ to $S^- s^+ = s^-$.

  ⇒ we get the matrices

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad S_3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2. Special Relativity (SR) — $SU(2)$ spinors

Constructing a spinor from the known number of eigenstates

- requiring the generators to be hermitian (for convenience)
  - we can write $S^{\pm} = S_1 \pm i S_2$
  - and recognise the commutation relations of the rotation group

$$[S_j, S_k] = i\epsilon_{jk\ell} S_\ell$$

$\Rightarrow$ the generators $S_k = \frac{1}{2}\sigma_k$ are given by the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- these Spin matrices act on 2d complex column vectors $\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$ $\Rightarrow$ (Weyl)spinors
  - each Weylspinor can also be written as a four parametric rotation ($\sigma^0 = \mathbf{1}_{2\times 2}$)

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = e^{i(\phi_0\sigma^0 + \phi_i\sigma^i)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = e^{i(\chi_0\sigma^0 + \chi_i\sigma^i)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ fundamental representation of the rotation group $SU(2)$

Rotations of Spinors

- with simple matrix multiplication we can see for the Pauli matrices:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{2\times2}$$

- So the finite rotation of a spinor around the $\hat{y}$-axis is

$$\begin{aligned} R[\theta] &= e^{i\theta S_y} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta \tfrac{1}{2}\sigma_y)^n = \sum_{n \text{ even}} \frac{1}{n!}(i\tfrac{\theta}{2})^n \sigma_y^n + \sum_{n \text{ odd}} \frac{1}{n!}(i\tfrac{\theta}{2})^n \sigma_y^n \\ &= \sum_n^{\infty} \frac{(-1)^n(\frac{\theta}{2})^{2n}}{(2n)!}(\sigma_y^2)^n + i\sum_n^{\infty} \frac{(-1)^n(\frac{\theta}{2})^{2n+1}}{(2n+1)!}(\sigma_y^2)^n \sigma_y \\ &= \cos\tfrac{\theta}{2} * 1_{2\times2} + i\sin\tfrac{\theta}{2}\sigma_y = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \end{aligned}$$

  – acting on the spinor $\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

⇒ spinors rotate only with half of the rotation angle θ

Invariants of the Poincaré group

- obviously $[ab,c] = a[b,c] + [a,c]b = abc - acb + acb - cab = abc - cab$

$$[P_\mu, P^2] = [P_\mu, P_\nu]P^\nu + P^\nu[P_\mu, P_\nu] = 0$$

and

$$\begin{aligned}[M_{\alpha\beta}, P^2] &= g^{\mu\nu}[M_{\alpha\beta}, P_\mu]P_\nu + g^{\mu\nu}P_\mu[M_{\alpha\beta}, P_\nu] \\ &= g^{\mu\nu} i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)P_\nu + g^{\mu\nu}P_\mu i(g_{\alpha\nu}P_\beta - g_{\beta\nu}P_\alpha) \\ &= -2i[P_\alpha, P_\beta] = 0 \ .\end{aligned}$$

$\Rightarrow$ $P^2 = m^2$ invariant is a consequence of the Poincaré algebra!

- Another invariant is W^2
  - with the Pauli-Lubanski vector $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}M_{\nu\rho}P_\lambda$

$$\begin{aligned}[P_\kappa, W^\mu] &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}([P_\kappa, M_{\nu\rho}]P_\lambda + M_{\nu\rho}[P_\kappa, P_\lambda]) \\ &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda} i(g_{\rho\kappa}P_\nu - g_{\nu\kappa}P_\rho)P_\lambda = 0\end{aligned}$$

$\Rightarrow$ Particles can be characterised simultaneously by the eigenvalues of P^2 and W^2

Invariants of the Poincaré group

- the spin vector W^μ is orthogonal to P_μ :

$$(P.W) = P^\mu \tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda} M^{\nu\rho} P^\lambda = 0$$

- For a particle at rest: $P_\mu = (m, 0)$ and $W_\mu = \frac{1}{2} m \epsilon_{\mu\nu\rho 0} M^{\nu\rho} = m(0, \vec{J})$
  - so $W^2 = -m^2 \vec{J}^2 = -m^2 s(s+1)$

  ⇒ eigenvalue of P^2 is m^2 and of W^2 is $m^2 s(s+1)$

- For a massless particle $P_\mu = (\eta, \eta, 0, 0)$
  - we have $P^2 = (P.W) = W^2 = 0$

    ⇒ eigenvalues of P^2 and W^2 are 0
  - but: $0 = \lambda^2 P^2 - 2\lambda(P.W) + W^2 = (\lambda P - W)^2$

  ⇒ therefore: $W^\mu = \lambda P^\mu$ with the helicity $\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$

    * λ depends on the representation (i.e. the spin) of the particle

⇒ Particles are characterised by mass and spin !

Investigating the Lorentz group

- distinguishing again boosts and rotations

$$K_i = M_{0i} = -M^{0i} \qquad \text{and} \qquad J_i = \tfrac{1}{2}\epsilon_{ijk}M^{jk} \ ,$$

the Lorentz algebra gives

$$[J_j, J_k] = i\epsilon_{jk\ell}J_\ell \ , \qquad [K_j, K_k] = -i\epsilon_{jk\ell}J_\ell \ , \qquad [J_j, K_k] = i\epsilon_{jk\ell}K_\ell$$

- defining

$$L_i = N_i = \tfrac{1}{2}(J_i + iK_i) \qquad \text{and} \qquad R_i = N_i^\dagger = \tfrac{1}{2}(J_i - iK_i)$$

one gets

$$[L_j, R_k] = 0 \ , \qquad [L_j, L_k] = i\epsilon_{jk\ell}L_\ell \ , \qquad [R_j, R_k] = i\epsilon_{jk\ell}R_\ell$$

- the Lorentz algebra is similar to $SU(2)_L \otimes SU(2)_R$!
- two invariants: $L_iL_i = n(n+1)$ and $R_iR_i = m(m+1)$
  - $J_i = L_i + R_i \quad \Rightarrow \quad$ spin $j = n + m$

Investigating the Lorentz group

- Parity leaves rotations invariant $J_i \xrightarrow{\mathsf{P}} J_i$, but flips boosts $K_i \xrightarrow{\mathsf{P}} -K_i$,
  $\Rightarrow$ $L_i \overset{\mathsf{P}}{\longleftrightarrow} R_i$, $(n,m) \overset{\mathsf{P}}{\longleftrightarrow} (m,n)$, $SU(2)_L \overset{\mathsf{P}}{\longleftrightarrow} SU(2)_R$

- Charge conjugation also interchanges $SU(2)_L \Leftrightarrow SU(2)_R$
  - like Parity

$\Rightarrow$ the combined transformation CP leaves $SU(2)_L$ and $SU(2)_R$ invariant
  - but it still includes mathematically a complex conjugation

- Time reversal T is an antiunitary transformation
  - it includes a complex conjugation

$\Rightarrow$ any quantum field theory

built from the representations of the Poincaré algebra

  - that means: scalars, spinors, vectors, . . .

has to be invariant under CPT

classifying particles

according to the eigenstates (n, m) of $SU(2)_L \otimes SU(2)_R$

- $(0, 0)$ is a scalar
- $(\frac{1}{2}, 0)$ is the χ_a left-handed Weyl-spinor
  - transforms with $\Lambda(\omega)_a{}^b = [e^{i\omega_{\alpha\beta}\sigma^{\alpha\beta}}]_a{}^b$
- $(0, \frac{1}{2})$ is the $\bar{\eta}^{\dot{a}}$ right-handed Weyl-spinor
  - transforms with $\Lambda(\omega)^{\dot{a}}{}_{\dot{b}} = [e^{i\omega_{\alpha\beta}\bar{\sigma}^{\alpha\beta}}]^{\dot{a}}{}_{\dot{b}}$
- $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ is $\Psi = \binom{\chi_a}{\bar{\eta}^{\dot{a}}}$, the Dirac-spinor
  - transforms with $\Lambda(\omega)^a{}_b = [e^{i\omega_{\alpha\beta}(-\frac{i}{4}[\gamma^\alpha, \gamma^\beta])}]^a{}_b$, with $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$
    - a and b go from 1 to 4, (3 and 4 representing the dotted indices)
- $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$ is $(\chi\sigma^\mu\bar{\eta}) = \chi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}$, the spin-1 four-vector

  ⇒ in that sense is the spinor the square root of the vector