# 1. Special Relativity (SR) — Introduction

## Lectures

- Introduction, Invariants
- Lorentz transformations
- Algebra of the Lorentz group

## Links

- Lecture notes by David Hogg: `http://cosmo.nyu.edu/hogg/sr/sr.pdf`
  - or: `http://www.tfk.ff.vu.lt/~garfield/WoP/sr.pdf`
- Tatsu Takeuchi: `http://www.phys.vt.edu/~takeuchi/relativity/notes/`

# 1. Special Relativity (SR) — Introduction

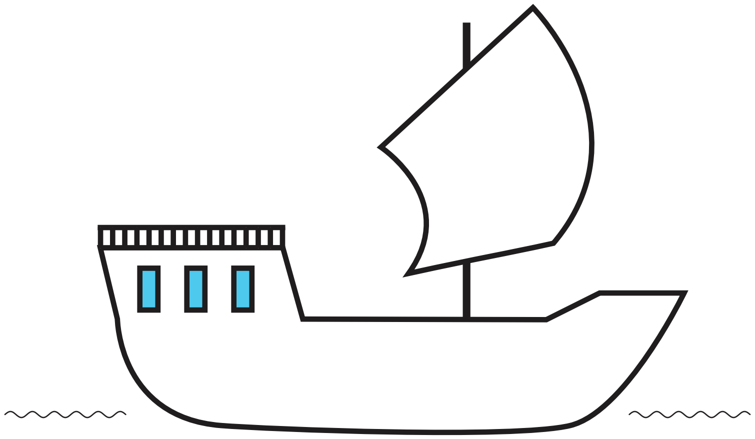
## History

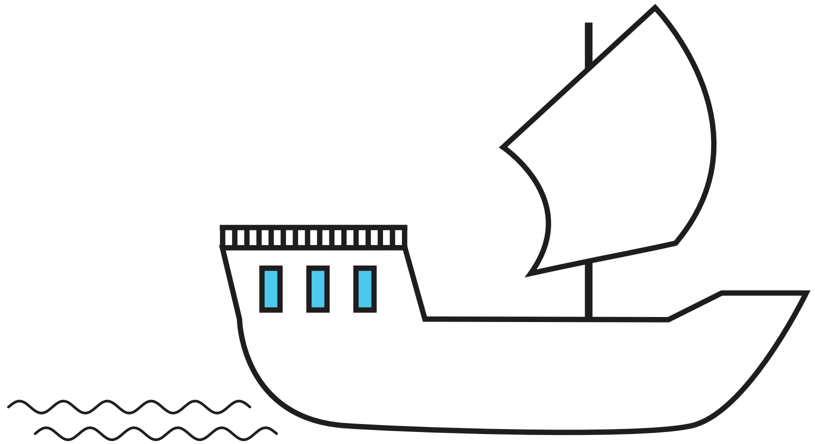
- 1632: Galileo Galilei describes the principle of relativity.
  - "Dialogue concerning the Two Chief World Systems"
- 1861: Maxwell's equations.
- 1887: Michelson-Morley experiment.
- 1889 / 1892: Lorentz – Fitzgerald transformation.
- 1905: Albert Einstein publishes the Theory of Special Relativity.
  - "On the Electrodynamics of Moving Bodies"
- 1908: Hermann Minkovsky introduces 4D space-time.

Galilean Invariance:

**Every physical theory should mathematically look the same to every inertial observer.**

- for Galileo it was the mechanics and kinematics:
  - water dropping down
  - throwing a ball or a stone
  - insects flying
  - jumping around





# 1. Special Relativity (SR) — Introduction

## Galilean Invariance / Galilean transformations: $t \to t'$, $\vec{x} \to \vec{x}'$

Two inertial observers, $O$ and $O'$,

- measure the same absolute time (i.e.: 1 second = 1 second′).
  - Time translations : $t' = t + \tau$, $\vec{x}' = \vec{x}$

    in index notation: $t' = t + \tau$, $x'_j = x_j$

- have at $t = 0$ a relative distance $\Delta\vec{r}$.
  - Spatial translations : $t' = t$, $\vec{x}' = \vec{x} + \Delta\vec{r}$

    in index notation: $t' = t$, $x'_j = x_j + \Delta r_j$

- have coordinate systems that are rotated by a relative rotation $\mathbf{R}$.
  - Rotations : $t' = t$, $\vec{x}' = \mathbf{R} \cdot \vec{x}$, where $\mathbf{R}$ is an orthogonal matrix

    in index notation: $t' = t$, $x'_j = \mathbf{R}_{jk} x_k = \sum_{k=1}^{3} \mathbf{R}_{jk} x_k$

- have a constant relative velocity $\vec{v}$, which can be zero, too.
  - Boosts : $t' = t$, $\vec{x}' = \vec{x} + \vec{v}t$

    in index notation: $t' = t$, $x'_j = x_j + v_j t$

## Galilean Group

- How the Galilean transformations act on a quantum mechanical state.

- What is a group?
  - a set with a binary operation:
  - an example is the set of numbers $\{0, 1, 2\}$ with the addition modulo 3 (i.e. taking only the remainder of the division by 3).

- Properties of a group
  - different transformations in the group do not give something that is outside the group.
  - two transformations in different order give either zero or another transformation.

- Each transformation depends on continuous parameters
  - The Galilean Group is a Lie Group.

## What's wrong with Galilean Invariance?

- Maxwell's equations describe the propagation of light depending on the electric permittivity and the magnetic permeability of the vacuum.
- If the vacuum is the same for every inertial observer, he has to measure the same speed of light regardless, who emitted it.
  - This is Einsteins second assumption!
- But then the addition of velocities described by the Galilean transformations are wrong.
- Lorentz transformations describe correctly the measurements done regarding the speed of light.
- Lorentz transformations include a transformation of the time, that the inertial observers measure.
- Absolut time is a concept, that is not able to describe nature.
  - That's wrong with the Galilean Invariance!

## 1. Special Relativity (SR) — Introduction

### Axioms of Special Relativity

- Every physical theory should look the same mathematically to every inertial observer.
- The speed of light in vacuum is independent from the movement of its emmitting body.

### Consequences

- The speed of light in vacuum is maximum speed for any information.
- The world has to be described by a 4D space-time: Minovsky space.
- The simplest object is a scalar (field): $\phi(x)$
  no structure except position and momentum.
- The next simplest object is a spinor (field): $\psi^{\alpha}(x)$
  a vector (field) can be described as a double-spinor.

## What are invariant objects?

- Objects that are the same for every inertial observer.
- Examples in 3D: rotations or translations
  - the distances $\ell$ between points: $\ell^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$.
  - the angle $\alpha$ between directions: $\cos\alpha = (\vec{a}\cdot\vec{b})/(|\vec{a}| * |\vec{b}|)$.
- In 4D Minkovsky space: $(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$.
  - The time $t$ is measured like spacial distances in meter.
  - The constant speed of light $c$ is used as the conversion factor between seconds and meters.
- Any scalar product of four-vectors in Minkovsky space:

$$(p.q) = p^\mu q^\nu g_{\mu\nu} = p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3 \quad .$$

  - This defines also the metric $g_{\mu\nu}$.

## Special scalar products

- Particles are usually described by their energy-momentum four-vector:

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p})$$

  - The mass of the particle is defined in its rest-frame: $\vec{p} = 0$.
  - There, the energy-momentum four-vector is $p^\mu = (m, 0)$.
  - Since $p^2 = (p \cdot p)$ is a scalar, it is the same in every frame.
  - In the rest-frame $p^2 = m^2$.
  - Therefore in every frame

$$m^2 = E^2 - \vec{p}^{\,2} \quad !$$

- This can be applied to collisions, too: $(p_1 + p_2)^2$ is constant.

## Lorentz transformations (LTs)

- relate the coordinate systems of two inertial observers.
- leave the "4-distance" invariant.
- assuming linearity, they can be written as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu} \quad .$$

  - These are called inhomogeneous Lorentz transformations $(\Lambda, a)$.

Homogeneous Lorentz transformations have $a^{\mu} = 0$.

- They leave scalar products invariant: $(p'.q') = (p.q)$.
- They describe 3 Rotations and 3 Boosts
  (cf. the Galilean transformations).
- They form a group: the Lorentz group
  - including the inhomogeneous LTs: the Poincaré group

## Lorentz transformations

- using the invariant scalar product $(p'.q') = (p.q)$

- the boost in $\hat{x}$ (1-direction) can be brought to the form

$$\Lambda^0_{\ 0} = \Lambda^1_{\ 1} = +\cosh\eta \qquad \Lambda^0_{\ 1} = \Lambda^1_{\ 0} = -\sinh\eta \ ,$$

  where $\eta$ is the "rapidity" of the boost.

- remembering $\tanh\eta = v/c := \beta$
  - and $\gamma = [1-\beta^2]^{-1/2} = [1-\tanh^2\eta]^{-1/2} = \cosh\eta$

- we get the Lorentz transformation in "conventional" form

$$t' = \gamma(t-\beta x) = \frac{E}{m}\left(t - \frac{p}{E}x\right) = m^{-1}(E\,t - p\,x)$$

$$x' = \gamma(x-\beta t) = \frac{E}{m}\left(x - \frac{p}{E}t\right) = m^{-1}(E\,x - p\,t)$$

## Lie groups and Lie algebras

- The $n \times n$ (complex) matrices form representations of Lie groups

- group multiplication is analytic $\Rightarrow$ expansion around unit element
  - unit element $e = \mathbf{1}_{n\times n}$
  - representation $T(g[\alpha]) = \exp[i\alpha_i X_i] \quad \Rightarrow \quad X_k = -i\frac{\partial T(g[\alpha])}{\partial \alpha_k}|_{\vec{\alpha}=0}$
  - generators $\{X_k\}$ span the representation of the Lie group

- the generators $\{X_k\}$ fulfill the Lie algebra $[X_j, X_k] = C_{jk}{}^{\ell} X_\ell$
  - with the antisymmetric structure constants $C_{jk}{}^{\ell} = -C_{kj}{}^{\ell}$
  - rank of the group: number of commuting generators
  - a Casimir operator commutes with all generators $\Rightarrow \propto e$

- the indices $i, j, k, \ell$ need not indicate single numbers!
  - for the generators we will have $X_i = X_{[mn]} = -X_{[nm]}$

# 3. Special Relativity (SR) – Algebra of the Poincaré group

Lorentz transformations ( like Galilean transformations )
consist of Boosts and Rotations

- a boost in $\hat{x}$ was done by

$$\Lambda(\eta)^{\mu}{}_{\nu} \;=\; \cosh\eta(\delta^{\mu}_{0}\delta^{0}_{\nu}+\delta^{\mu}_{1}\delta^{1}_{\nu})-\sinh\eta(\delta^{\mu}_{0}\delta^{1}_{\nu}+\delta^{\mu}_{1}\delta^{0}_{\nu})+\delta^{\mu}_{2}\delta^{2}_{\nu}+\delta^{\mu}_{3}\delta^{3}_{\nu}$$

- a rotation between $\hat{y}$ and $\hat{z}$ can be done by

$$\Lambda(\theta)^{\mu}{}_{\nu} = \delta^{\mu}_{0}\delta^{0}_{\nu}+\delta^{\mu}_{1}\delta^{1}_{\nu}+\cos\theta(\delta^{\mu}_{2}\delta^{2}_{\nu}+\delta^{\mu}_{3}\delta^{3}_{\nu})-\sin\theta(\delta^{\mu}_{2}\delta^{3}_{\nu}-\delta^{\mu}_{3}\delta^{2}_{\nu})$$

- we obtain the generators for boosts with $-i\frac{\partial\Lambda(\eta)^{\mu}{}_{\nu}}{\partial\eta}|_{\eta=0} =$

$$-i\sinh\eta(\delta^{\mu}_{0}\delta^{0}_{\nu}+\delta^{\mu}_{1}\delta^{1}_{\nu})+i\cosh\eta(\delta^{\mu}_{0}\delta^{1}_{\nu}+\delta^{\mu}_{1}\delta^{0}_{\nu})|_{\eta=0} = i(\delta^{\mu}_{0}\delta^{1}_{\nu}+\delta^{\mu}_{1}\delta^{0}_{\nu})$$

- we obtain the generators for rotations with $-i\frac{\partial\Lambda(\theta)^{\mu}{}_{\nu}}{\partial\theta}|_{\theta=0} =$

$$+i\sin\theta(\delta^{\mu}_{2}\delta^{2}_{\nu}+\delta^{\mu}_{3}\delta^{3}_{\nu})+i\cos\theta(\delta^{\mu}_{2}\delta^{3}_{\nu}-\delta^{\mu}_{3}\delta^{2}_{\nu})|_{\theta=0} = i(\delta^{\mu}_{2}\delta^{3}_{\nu}-\delta^{\mu}_{3}\delta^{2}_{\nu})$$

- The other boosts go in $\hat{y}$ or $\hat{z}$ direction: $i(\delta^\mu_0 \delta^i_\nu + \delta^\mu_i \delta^0_\nu)$,
  or with the indices $0i$ down: $(M_{0i})^\mu{}_\nu = i(\delta^\mu_0(-g_{i\nu}) + \delta^\mu_i g_{0\nu})$.
- The other rotations go in $\hat{x}\hat{y}$ or $\hat{x}\hat{z}$ direction: $i(\delta^\mu_j \delta^k_\nu - \delta^\mu_k \delta^j_\nu)$,
  or with the indices $jk$ up: $(M_{jk})^\mu{}_\nu = i(\delta^\mu_j(-g_{k\nu}) - \delta^\mu_k(-g_{j\nu}))$.
- both generators have now the same form: $(M_{\alpha\beta})^\mu{}_\nu = -i(\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu})$
- with $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ we get

$$\Lambda(\omega)^\mu{}_\nu = \exp[i(M_{\alpha\beta}\omega^{\alpha\beta})^\mu{}_\nu] = \exp[(\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu})\omega^{\alpha\beta}]$$

- these generators fulfill the Lie algebra of the Lorentz group:

$$[M_{\alpha\beta}, M_{\gamma\delta}]^\mu{}_\nu = i(g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\delta}M_{\beta\gamma} + g_{\beta\delta}M_{\alpha\gamma})^\mu{}_\nu$$

- unifying time and spatial translations $P_\mu = (H, P_i)$
- we get the rest of the Poincaré algebra:

$$[P_\mu, P_\nu] = 0 \qquad \text{and} \qquad [M_{\alpha\beta}, P_\mu] = i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)$$

Weyl Spinors form the fundamental representation of $SU(2)$

- They are the simplest spinors: they have only two components
  - like the electron in the Stern-Gerlach experiment
- Lorentz transformations have to act two-dimensional
  - a representation can be formed by the Pauli matrices
- the generators have to have the same form: $(M_{\alpha\beta})^{\mu}{}_{\nu}$
  - with $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ describing the LTs as before
  - but $\mu, \nu = 1, 2$ are now spinor indices
  
  $\Rightarrow$ $(M_{\alpha\beta})^{\mu}{}_{\nu}$ has to be a linear combination of Pauli matrices
- introducing $(\sigma^{\mu})_{a\dot{a}} = (\sigma^0, \sigma^i)_{a\dot{a}}$ and $(\bar{\sigma}^{\mu})^{\dot{a}a} = (\sigma^0, -\sigma^i)^{\dot{a}a}$
  - where $\sigma^0 = 1_{2\times 2}$ and $\sigma^i$ are the Pauli matrices
  - we can write the generators as

$$(g^{\alpha\gamma}g^{\beta\delta}M_{\gamma\beta})_a{}^b = (\sigma^{\alpha\beta})_a{}^b = -\frac{i}{4}(\sigma^{\alpha}\bar{\sigma}^{\beta} - \sigma^{\beta}\bar{\sigma}^{\alpha})_a{}^b$$

and

$$(g^{\alpha\gamma}g^{\beta\delta}M_{\alpha\beta})^{\dot{a}}{}_{\dot{b}} = (\bar{\sigma}^{\alpha\beta})^{\dot{a}}{}_{\dot{b}} = -\frac{i}{4}(\bar{\sigma}^{\alpha}\sigma^{\beta} - \bar{\sigma}^{\beta}\sigma^{\alpha})^{\dot{a}}{}_{\dot{b}}$$