Spontaneous broken gauge symmetry

- consider a complex scalar doublet $\phi = (\phi_1, \phi_2)^\top$, so $\phi^\dagger = (\phi_1^*, \phi_2^*)$
- the corresponding $SU(2)$ gauge field $\widehat{W}_\mu = \sum_{i=a}^3 W_\mu^a \frac{1}{2}\sigma^a$
  - with the covariant derivative $D_\mu = \partial_\mu - ig\widehat{W}_\mu$
  - the field strength $\widehat{W}_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu \widehat{W}_\nu - \partial_\nu \widehat{W}_\mu - ig[\widehat{W}_\mu, \widehat{W}_\nu]$
  - or with $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$: $\quad W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\epsilon^{abc} W_\mu^b W_\nu^c$
- the gauge transformation with $\widehat{\alpha} = \frac{1}{2}\sigma^j \alpha^j$

$$\phi \overset{\alpha}{\to} \phi' = e^{i\widehat{\alpha}}\phi \qquad \widehat{W}_\mu \overset{\alpha}{\to} \widehat{W}'_\mu = \widehat{W}_\mu + \frac{1}{g}[D_\mu, \widehat{\alpha}]$$

or infinitesimal

$$\delta_\alpha \phi = \phi' - \phi = i(\tfrac{1}{2}\sigma^a \phi)\alpha^a \qquad \delta_\alpha W_\mu^a = W_\mu^{\prime a} - W_\mu^a = \frac{1}{g}\partial_\mu \alpha^a + \epsilon^{abc} W_\mu^b \alpha^c$$

- the $SU(2)$ invariant Lagrangian is like scalar QED:

$$\mathcal{L} = (D^\mu \phi)^\dagger (D_\mu \phi) - \frac{1}{4} W_{\mu\nu}^a W^{a\,\mu\nu} - \frac{1}{2}\mu^2 \phi^\dagger \phi - \frac{1}{4}\lambda(\phi^\dagger \phi)^2$$

Spontaneous broken gauge symmetry

- The vacuum is the state of minimal energy
- for $\mu^2 < 0$, this minimum is at $|\phi| := \sqrt{|\phi_1|^2 + |\phi_2|^2} = \sqrt{\frac{-\mu^2}{\lambda}} =: v$
- So one of the 4 parameters of the scalar field is fixed at the vaccum:
  - ϕ aquires a vacuum expectation value
  - how the 4 degrees of freedom of ϕ are fixed is a gauge choice!
    - we choose $\phi = \begin{pmatrix} \chi^+ \\ v + (h + i\chi_3)/\sqrt{2} \end{pmatrix} = v \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \chi$
- so we have $\phi^\dagger \phi = v^2 + \sqrt{2} v h + |\chi|^2$ with $|\chi|^2 = \frac{1}{2}(h^2 + \chi_3^2) + \chi^+ \chi^-$, $D_\mu \phi = D_\mu \chi - igv\widehat{W}_\mu (0,1)^\top$, and the potential part

$$
\begin{aligned}
V(\chi) &= \tfrac{1}{2}\mu^2(v^2 + \sqrt{2}vh + |\chi|^2) + \tfrac{1}{4}\lambda(v^2 + \sqrt{2}vh + |\chi|^2)^2 \\
&= \tfrac{1}{2}\mu^2 v^2 + \tfrac{\lambda}{4}v^4 + \tfrac{1}{2}(\mu^2 + \lambda v^2)(\sqrt{2}vh + |\chi|^2) + \tfrac{1}{4}\lambda(\sqrt{2}vh + |\chi|^2)^2 \\
&= \tfrac{1}{4}\mu^2 v^2 - \tfrac{1}{2}\mu^2 h^2 + \tfrac{\sqrt{2}}{2}\lambda v h |\chi|^2 + \tfrac{\lambda}{4}|\chi|^4
\end{aligned}
$$

Spontaneous broken gauge symmetry

- Note that the potential is no longer zero, when there is no field!
- the real part h got a mass term with a mass $m_h^2 = -\mu^2 > 0$
- the imaginary χ_3 and charged parts χ^- have no mass: $m_\chi^2 = 0$

  $\Rightarrow$ they became Goldstone Bosons !
- The covariant derivative part $(D_\mu \phi)^\dagger (D^\mu \phi)$ gives

$$
\begin{aligned}
&= (D_\mu \chi - igv\widehat{W}_\mu (0,1)^\top)^\dagger (D^\mu \chi - igv\widehat{W}^\mu (0,1)^\top) \\
&= (D_\mu \chi)^\dagger (D^\mu \chi) + igv(0,1)\widehat{W}_\mu^\dagger D^\mu \chi - igv(D_\mu \chi)^\dagger \widehat{W}^\mu (0,1)^\top \\
&\quad + g^2 v^2 (0,1)\widehat{W}_\mu^\dagger \widehat{W}^\mu (0,1)^\top
\end{aligned}
$$

- the last term gives a mass term for the gauge field: $g^2 v^2 W_\mu^a W^{a\,\mu}$

this is the Higgs effect !

broken global symmetries

- for exact symmetries:
  - there are Nambu-Goldstone bosons, which are massless
- For approximate symmetries:
  - there are Pseudo-Goldstone bosons, which have a small mass

unbroken local symmetry or gauge symmetry

- gauge bosons are massless
- each gauge boson has only two degrees of freedom:
  - transverse polarisations

spontaneous broken gauge symmetry

- gauge bosons along the broken directions aquire a mass
- each such gauge boson has now three degrees of freedom:
  - transverse polarisations and a longitudinal polarisation
  - the Goldstone bosons are "eaten up"
  - and give the additional degree of freedom

The Standard Model gauge groups

- $U(1)_Y = U(1)_{\text{Hypercharge}}$ with the generator Y
- $SU(2)_L = SU(2)_{\text{weak}}$ with the generators $T^i = \frac{1}{2}\sigma^i$
- $SU(2)_L \times U(1)_Y$ is spontaneously broken to $U(1)_{\text{em}} = U(1)_{\text{electro-magnetic}}$
- $SU(3) = SU(3)_{\text{color}}$ with the generators $F^i = \frac{1}{2}\lambda^i$
- $SU(3)_{\text{color}}$ is unbroken

Particles of the Standard Model

- all particles except the gauge bosons are charged under $U(1)_Y$
- all left handed fermions, the higgs and the $SU(2)$-gauge bosons are carged under $SU(2)_L$
- only quarks and gluons are charged under $SU(3)$
- only electrically charged particles transform under $U(1)_{\text{em}}$

$SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$

- the higgs is charged under $SU(2)_L$ and $U(1)_Y$
- its vacuum expectation value (vev) breaks both gauge groups
- the vev is electrically neutral, so it does not break $U(1)_{\text{em}}$
- the mixing between the $SU(2)$- and $U(1)$-gauge bosons is described by the weak mixing angle or Weinberg angle θ_w with $\sin^2\theta_w \sim 0.23$
- with $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we can write

  $Y = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$, where y is the Hypercharge of the corresponding particle.
  - that is $y = \frac{1}{2}$ in the case of the Higgs.

$SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$

- writing the $U(1)_Y$ gauge field as B_μ
- the covariant derivative of the complex scalar doublet becomes

$$D_\mu \phi = \left[\partial_\mu - i g' B_\mu \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} - i g \frac{1}{2} \begin{pmatrix} W_\mu^3 & W_\mu^1 - i W_\mu^2 \\ W_\mu^1 + i W_\mu^2 & -W_\mu^3 \end{pmatrix} \right] \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

- parameterizing the complex scalar doublet as $\phi = (\chi^+, v + \frac{1}{\sqrt{2}}(h + i\chi^0))^\top$
  - this will give the canonical normalisation of the real scalar field h
- $W_\mu^\pm := \frac{1}{\sqrt{2}}(W_\mu^1 \mp i W_\mu^2)$, $Z_\mu := \frac{g}{g_z} W_\mu^3 - \frac{g'}{g_z} B_\mu$, and $A_\mu := \frac{g'}{g_z} W_\mu^3 + \frac{g}{g_z} B_\mu$
- $g_z^2 = g^2 + g'^2$, $g_e = \frac{g' g}{g_z}$, and $\sin\theta_w = \frac{g'}{g_z} =: s_w$
- gives the covariant derivative

$$D_\mu \phi = \left[\partial_\mu - i \begin{pmatrix} g_e A_\mu + g_z(\frac{1}{2} - s_w^2) Z_\mu & \frac{g}{\sqrt{2}} W_\mu^+ \\ \frac{g}{\sqrt{2}} W_\mu^- & -\frac{g_z}{2} Z_\mu \end{pmatrix} \right] \begin{pmatrix} \chi^+ \\ v + \frac{1}{\sqrt{2}}(h + i\chi^0) \end{pmatrix}$$

$SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$

- and $(D^\mu\phi)^\dagger(D_\mu\phi)$

$$= \left|\partial_\mu\chi^+ - i[g_e A_\mu + g_z(\tfrac{1}{2} - s_w^2)Z_\mu]\chi^+ - i\tfrac{g}{\sqrt{2}}W_\mu^+[v + \tfrac{1}{\sqrt{2}}(h + i\chi^0)]\right|^2$$

$$+ \left|\partial_\mu\tfrac{1}{\sqrt{2}}(h + i\chi^0) - i\tfrac{g}{\sqrt{2}}W_\mu^-\chi^+ + i\tfrac{g_z}{2}Z_\mu[v + \tfrac{1}{\sqrt{2}}(h + i\chi^0)]\right|^2$$

gives the bilinear terms

$$\left|\partial_\mu\chi^+ - i\tfrac{gv}{\sqrt{2}}W_\mu^+\right|^2 + \tfrac{1}{2}\left|\partial_\mu(h + i\chi^0) + i\tfrac{g_z v}{\sqrt{2}}Z_\mu\right|^2$$

$$= \tfrac{1}{2}(\partial^\mu h)(\partial_\mu h) + (\partial^\mu\chi^-)(\partial_\mu\chi^+) + \tfrac{1}{2}(\partial^\mu\chi^0)(\partial_\mu\chi^0)$$

$$+ \tfrac{igv}{\sqrt{2}}W_\mu^-\partial^\mu\chi^+ - \tfrac{igv}{\sqrt{2}}W_\mu^+\partial^\mu\chi^- + \tfrac{g_z v}{\sqrt{2}}Z_\mu\partial^\mu\chi^0 + \tfrac{g^2v^2}{2}W_\mu^+W^{-\mu} + \tfrac{g_z^2v^2}{4}Z_\mu Z^\mu$$

- W and Z bosons have different masses: $m_W^2 = \frac{g^2v^2}{2}$ and $m_Z^2 = \frac{g_z^2v^2}{2}$
  - their ratio is (on tree level) $\cos\theta_w = \frac{m_W}{m_Z}$

$SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$

- the higgs field h has it's canonical kinetic term
  - the higgs mass comes from the potential; with
    $\phi^\dagger\phi = \chi^-\chi^+ + |v + \frac{1}{\sqrt{2}}(h + i\chi^0)|^2 = v^2 + \sqrt{2}vh + \frac{1}{2}(h^2 + \chi^{0\,2}) + \chi^-\chi^+$:

$$
\begin{aligned}
V(h,\chi) &= +\tfrac{1}{2}(\mu^2 + \tfrac{1}{2}\lambda v^2)v^2 + \tfrac{1}{2}(\mu^2 + \lambda v^2)(\sqrt{2}vh + \tfrac{1}{2}\chi^{0\,2} + \chi^-\chi^+) \\
&\quad + \tfrac{1}{2}(\tfrac{1}{2}\mu^2 + \tfrac{3}{2}\lambda v^2)h^2 + \tfrac{\lambda}{2\sqrt{2}}vh^3 + \tfrac{\lambda}{16}h^4 \\
&\quad + \tfrac{\lambda}{4}(2\sqrt{2}vh + h^2)(\tfrac{1}{2}\chi^{0\,2} + \chi^-\chi^+) + \tfrac{\lambda}{4}(\tfrac{1}{2}\chi^{0\,2} + \chi^-\chi^+)^2 \\
&= \tfrac{1}{4}\mu^2v^2 - \tfrac{1}{2}\mu^2h^2 + \tfrac{\lambda v}{2\sqrt{2}}h^3 + \tfrac{\lambda}{16}h^4 \\
&\quad + \tfrac{\lambda}{4}(2\sqrt{2}vh + h^2)(\tfrac{1}{2}\chi^{0\,2} + \chi^-\chi^+) + \tfrac{\lambda}{4}(\tfrac{1}{2}\chi^{0\,2} + \chi^-\chi^+)^2
\end{aligned}
$$

  - $m_h^2 = -\mu^2 = \lambda v^2 > 0$
- the Goldstone bosons still have masses $m_{\chi^0}^2 = m_{\chi^\pm}^2 = 0$

Gauge fixing $SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$

a convenient and often used gauge choice is the R_ξ-gauge with

$$G^\pm = \partial^\mu W^\pm_\mu - \xi_w m_W \chi^\pm \quad \text{and} \quad G^0 = \partial^\mu Z_\mu - \xi_z m_Z \chi^0$$

and

$$\mathcal{L}_{\text{gaugefixing}} \simeq \frac{1}{\xi_w}|G^\pm|^2 + \frac{1}{2\xi_z}(G^0)^2$$

- the first part of the squares, $\frac{1}{\xi_w}(\partial^\mu W^+_\mu)(\partial^\nu W^-_\nu)$ or $\frac{1}{2\xi_z}(\partial^\mu Z_\mu)^2$ allows to calculate the propagator:

$$\langle V(k)V(-k)\rangle = \frac{-i}{k^2 - m_V^2}\left(g^{\mu\nu} - \frac{(1-\xi_V)k^\mu k^\nu}{k^2 - \xi_V m_V^2}\right) \quad \text{, with} \quad V_\mu = W^\pm_\mu, Z_\mu$$

- the second part adds with bilinear terms of $(D^\mu\phi)^\dagger(D_\mu\phi)$ to form total derivatives:

$$\partial^\mu\left[\frac{igv}{\sqrt{2}}(W^-_\mu\chi^+ - W^+_\mu\chi^-) + \frac{g_z v}{\sqrt{2}} Z_\mu \chi^0\right]$$

- the third part gives mass terms for the Goldstone bosons:

$$\xi_w m_W^2 \chi^+\chi^- \quad \text{and} \quad \frac{1}{2}\xi_z m_Z^2 (\chi^0)^2$$

  – these masses, $\xi_w m_W^2$ and $\xi_z m_Z^2$, are gauge dependent

  $\Rightarrow$ they cannot describe physical particles

Gauge fixing $SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$

another possible gauge is the unitary gauge ($\xi_V \to \infty$)

- the higgs is parameterized as $\phi = U(0, v + \frac{1}{\sqrt{2}} h)^\top$ with $U \in SU(2)$
- U has then the gauge transformation $U \to U' = U_{\text{gauge}} U$
- the gauge choice $U_{\text{gauge}} = U^{-1}$ gets rid of the Goldstone bosons.
- but the vector boson propagator

$$\langle V(k) V(-k) \rangle = \frac{-i}{k^2 - m_V^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_V^2} \right) \quad \text{, with} \quad V_\mu = W_\mu^\pm, Z_\mu$$

does not fall off for $k^2 \to \infty$

$\Rightarrow$ Problems in the high energy limit

$\Rightarrow$ Problems with loops !

Different gauge choices for solving different problems !

gauge transformations on fermions

$SU(3)_{\text{color}}$ gauge transformations only affect the color of quarks and gluons:

- the gauge transformation can be written as a matrix in color space:

$$U_s := U^{rs}_{(\varphi_s)} = (e^{ig_s T^a \varphi_s^a})^{rs}$$

- quarks transform like a normal vector in color space:

$$q^r \to q'^r = U^{rs}_{(\varphi_s)} q^s \qquad \text{or simply} \quad q \to q' = U_s q$$

- antiquarks transform with the inverse transformation: $\bar{q} \to \bar{q}' = \bar{q} U_s^{-1}$
- gluons transform with both: $G^a_\mu T^a \to G'^a_\mu T^a = U_s G^a_\mu T^a U_s^{-1} + \frac{i}{g_s}(\partial_\mu U_s) U_s^{-1}$
  - that gives for the change of the gluon field $\delta G^a_\mu = G'^a_\mu - G^a_\mu$

$$\delta G^a_\mu = -\partial_\mu \varphi^a_s + g_s f^{abc} G^b_\mu \varphi^c_s = -D^{ac}_\mu \varphi^c_s$$

  - from from the gauge fixing function $F^a = \partial^\mu G^a_\mu$ one obtains the Faddeev-Popov determinant

$$\Delta_g[G^a_\mu] = \text{Det}\left[\frac{\delta F^a}{\delta \varphi^c_s}\right] = \text{Det}\left[-\partial^\mu D^{ac}_\mu\right] = \int \mathcal{D}b\, \mathcal{D}c\, (\partial^\mu b^a)[D^{ac}_\mu c^c] = \int \mathcal{D}b\, \mathcal{D}c\, (\partial^\mu b^a)[\partial_\mu \delta^{ac} - g_s f^{abc} G^b_\mu] c^c$$

  - ghosts η are in the adjoint representation: $\eta^a T^a \to \eta'^a T^a = U_s \eta^a T^a U_s^{-1} \quad \Rightarrow \quad \delta\eta^a = f^{abc}\eta^b \varphi^c_s$

- the QCD Lagrangian is invariant under these gauge transformations

$$\mathcal{L} \to \mathcal{L}' = \bar{q}'(i\not{D}' - m)q' = \bar{q} U_s^{-1}(i U_s \not{D} U_s^{-1} - m) U_s q = \bar{q}(i\not{D} - m)q = \mathcal{L}$$

$\Rightarrow$ because left-handed and right-handed quarks transform in the same way

$SU(2)_L \times U(1)_Y$ gauge transformations on fermions

$SU(2)_L$ gauge transformations affect only left-handed fermions and the Higgs doublet

$$\ell = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad , \qquad q = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad , \qquad \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

- the gauge transformation can be written as a 2×2-matrix
$$U_L := U^{jk}_{(\varphi_L)} = (e^{igT^a\varphi^a_L})^{jk} = e^{\frac{i}{2}g\sigma^a\varphi^a_L}$$
- the doublets $2_L = \ell, q, \phi$ transform like a normal vectors: $2_L \to 2'_L = U_L 2_L$
- antidoublets transform with the inverse transformation: $\bar{2}_L \to \bar{2}'_L = \bar{2}_L U_L^{-1}$
- $SU(2)_L$ gauge bosons transform with both: $W^a_\mu T^a \to W'^a_\mu T^a = U_L W^a_\mu T^a U_L^{-1} + \frac{i}{g}(\partial_\mu U_L)U_L^{-1}$
  - that gives $\delta W^a_\mu = W'^a_\mu - W^a_\mu = -\partial_\mu \varphi^a_L + g\epsilon^{abc}W^b_\mu \varphi^c_L = -D^{ac}_\mu \varphi^c_L$

$U(1)_Y$ gauge transformation affects all fermions and the Higgs doublet

$$\ell \ , \quad e_R \quad , \qquad q \ , \quad u_R \ , \quad d_R \quad , \qquad \phi$$

- but left-handed and right-handed fermions differently
- the gauge transformation can be written as a simple phase transformation $U_Y = e^{ig'y\varphi_Y}$
  - y_f is the Hypercharge of the field f on which the $U(1)_Y$ transformation acts:

$$f \to f' = e^{ig'y_f\varphi_Y} f \quad \text{and} \qquad \bar{f} \to \bar{f}' = e^{-ig'y_f\varphi_Y} \bar{f}$$

$SU(2)_L \times U(1)_Y$ gauge transformations on fermions

- the Lagrangian has to be invariant under $SU(2)_L \times U(1)_Y$
  - but we cannot even construct an invariant mass term!
    - ∗ as the mass term mixes left-handed and right-handed fermions

  ⇒ The Standard Model is a mass-less chiral theory

- to build a Lagrangian invariant under $SU(2)_L \times U(1)_Y$
  - we first combine doublets in an invariant way:

$$(\bar{\nu}_\ell, \bar{\ell}_L) \cdot \phi = \bar{\nu}_\ell \phi_1 + \bar{\ell}_L \phi_2 \qquad (\bar{u}_L, \bar{d}_L) \cdot \phi = \bar{u}_L \phi_1 + \bar{d}_L \phi_2$$

and

$$\det\left[(\bar{u}_L, \bar{d}_L)^\top, \phi^*\right] = \det\begin{vmatrix} \bar{u}_L & \phi_1^* \\ \bar{d}_L & \phi_2^* \end{vmatrix} = \bar{u}_L \phi_2^* - \bar{d}_L \phi_1^*$$

    - ∗ these products of fields are fermionic and transform under $U(1)_Y$
  - the right-handed singlets have to guarantee $U(1)_Y$ conservation:

$$(\bar{\nu}_\ell \phi_1 + \bar{\ell}_L \phi_2)\ell_R \qquad (\bar{u}_L \phi_1 + \bar{d}_L \phi_2) d_R \qquad (\bar{u}_L \phi_2^* - \bar{d}_L \phi_1^*) u_R$$

    - ∗ this restricts the possible values of y_f for the right-handed singlets

$SU(2)_L \times U(1)_Y$ gauge transformations on fermions

- the three gauge invariant terms

$$(\bar{\nu}_\ell \phi_1 + \bar{\ell}_L \phi_2)\ell_R \qquad (\bar{u}_L \phi_1 + \bar{d}_L \phi_2) d_R \qquad (\bar{u}_L \phi_2^* - \bar{d}_L \phi_1^*) u_R$$

  describe couplings of the mass less fermions to the Higgs doublet
  - they can have arbitrary coupling constants
  - considering that the fermions come in three generations

    $\Rightarrow$ coupling constants can be 3×3 matrices: Yukawa matrices

- the matrices for u_R and d_R are independent
  - diagonalization (singular value decomposition) of Y_u and Y_d differs

    $\Rightarrow$ leads to mixing between generations and allows for CP-violation

- introducing the vev $\phi_1 \to 0$, $\phi_2 \to v$ and "masses"

$$m_\ell = vY_\ell \quad , \qquad m_d = vY_d \quad , \quad \text{and} \qquad m_u = vY_u$$

  we get the fermion mass terms:

$$\mathcal{L}_{\text{mass}} = m_\ell \bar{\ell}_L \ell_R + m_d \bar{d}_L d_R + m_u \bar{u}_L u_R$$