Physics of the renormalized gauge boson propagator, continued

- the photon propagator describes the interaction of charges
  - one can compare the renormalized 1-loop result with the tree level
  - again we have to consider elastic scattering
    - the scattered particles stay the same
    - defining the currents $J_e^\mu = \bar{u}(q_e)\gamma^\mu u(p_e)$ and $J_p^\mu = \bar{u}(q_p)\gamma^\nu u(p_p)$
- the amplitude for elastic $e^- p$ scattering:

$$i(2\pi)^4\delta^4(p_e + p_p - q_e - q_p)\mathcal{M}^{\text{QFT}}$$

$$= \int \frac{d^4k}{(2\pi)^4}(-igQ_e)(2\pi)^4\delta^4(p_e + k - q_e)J_e^\mu i\Delta_{\mu\nu}(k)(-igQ_p)(2\pi)^4\delta^4(p_p - k - q_p)J_p^\nu$$

$$= i(2\pi)^4\delta^4(p_{\text{in}} - q_{\text{out}})g^2 Q_e Q_p J_e^\mu \left[\frac{-ig_{\mu\nu}}{k^2[1 - \bar{\Pi}_\gamma^{[2]}(k)]} + \frac{ik_\mu k_\nu}{k^4}\left(\frac{1}{1 - \bar{\Pi}_\gamma^{[2]}(k)} - \xi\right)\right] J_p^\nu$$

  - with $k^\mu = q_e^\mu - p_e^\mu = p_p^\mu - q_p^\mu$
- putting momenta from the gauge dependent part into the currents:

$$J_e^\mu k_\mu = \bar{u}(q_e)\gamma^\mu u(p_e)k_\mu = \bar{u}(q_e)(\not{q}_e - \not{p}_e)u(p_e) = \bar{u}(q_e)(m_e - m_e)u(p_e) = 0$$

$\Rightarrow$ current conservation

Physics of the renormalized gauge boson propagator, continued

- the one-loop renormalized scattering amplitude

$$\mathcal{M}^{[2]} = Q_e Q_p\, g^2 J_e^\mu \frac{-ig_{\mu\nu}}{k^2[1-\bar{\Pi}_\gamma^{[2]}(k)]} J_p^\nu \overset{\mathcal{O}(g^2)}{=} Q_e Q_p\, g^2 [1+\bar{\Pi}_\gamma^{[2]}(k)] J_e^\mu \frac{-ig_{\mu\nu}}{k^2} J_p^\nu$$

  can be compared with the tree level

$$\mathcal{M}^{[0]} = Q_e Q_p\, g^2 J_e^\mu \frac{-ig_{\mu\nu}}{k^2} J_p^\nu$$

  ⇒ motivates the definition of a running coupling constant

$$g^2(Q^2) := g^2[1-\bar{\Pi}_\gamma^{[2]}(Q^2)]^{-1}$$

  – defining the energy scale $Q^2 = |k^2|$
    * in elastic scattering k^μ is space-like in the CM-frame

    ⇒ so $k^2 < 0$, and hence $Q^2 = -k^2$

- evaluating $\bar{\Pi}_\gamma^{[2]}(Q^2)$ for $Q^2 = -q^2 \gg m^2$
  – $\bar{\Pi}$ was given by $\bar{\Pi}_\gamma(q^2) = \Pi_\gamma(q^2) - (Z_3 - 1)$
  – the renormalization condition was $\lim_{q^2\to 0} \bar{\Pi}_\gamma(q^2) = 0$

  ⇒ $(Z_3 - 1) = \Pi_\gamma(0)$ and $\bar{\Pi}_\gamma(q^2) = \Pi_\gamma(q^2) - \Pi_\gamma(0)$

Physics of the renormalized gauge boson propagator, continued

- we calculated the regularized gauge boson self energy

$$\Pi_\gamma^{[2]}(q^2) = -\frac{8g^2}{(4\pi)^2}\int_0^1 dx \int \frac{d^4p}{i\pi^2}\frac{x(1-x)}{[p^2+x(1-x)(-Q^2)-m^2+i\epsilon]^2}$$

$$= \frac{8g^2}{(4\pi)^{D/2}}\int_0^1 dx\, x(1-x)\left(\frac{2}{4-D}+\gamma_{\text{E.M.}}+\ln[m^2]+\ln[1+x(1-x)\frac{Q^2}{m^2}]+\mathcal{O}(4-D)\right)$$

  - in the limit $D\to 4$ we get (with $Q^2=-q^2$)

$$\Pi_\gamma^{[2]}(0) = \frac{8g^2}{(4\pi)^{D/2}}\int_0^1 dx\, x(1-x)\left(\frac{2}{4-D}+\gamma_{\text{E.M.}}+\ln[m^2]\right) = Z_3 - 1$$

$$\Rightarrow \quad \hat{\Pi}_\gamma^{[2]}(q^2) = \frac{8g^2}{(4\pi)^{D/2}}\int_0^1 dx\, x(1-x)\ln[1+x(1-x)\frac{Q^2}{m^2}] \quad \ldots \text{ calculate with Mathematica}$$

$$\approx \frac{8g^2}{(4\pi)^2}\left(\frac{1}{6}\ln[\frac{Q^2}{m^2}]-\frac{5}{18}+\frac{m^2}{Q^2}+\mathcal{O}([\frac{m^2}{Q^2}]^2)\right) = \frac{\alpha}{3\pi}\left(\ln[\frac{Q^2}{m^2}]-\frac{5}{3}+\mathcal{O}(\frac{m^2}{Q^2})\right)$$

- this gives the energy dependent fine structure constant ($\alpha := \frac{g^2}{4\pi}$)

$$\alpha(Q^2) = \frac{\alpha}{1-\hat{\Pi}_\gamma^{[2]}(Q^2)} = \frac{\alpha}{1-\frac{\alpha}{3\pi}(\ln[\frac{Q^2}{m^2}]-\frac{5}{3})} = \frac{\alpha}{1-\frac{\alpha}{3\pi}\ln[\frac{Q^2}{Am^2}]}$$

  - with $A = e^{5/3}$ running coupling constant

Changing the scales of measurements

- renormalization conditions give the energy scale of our measurement
  - changing the scale in the renormalization condition from 0 to μ^2
  - changes the value of the counter terms:

$$
\begin{aligned}
Z_3(0) &= 1 + \Pi_\gamma^{[2]}(0) \to Z_3(\mu^2) = 1 + \Pi_\gamma^{[2]}(\mu^2) \\
&= \frac{8g^2}{(4\pi)^{D/2}} \int_0^1 dx\, x(1-x) \left(\frac{2}{4-D} + \gamma_{\text{E.M.}} + \ln[m^2 + x(1-x)\mu^2] \right)
\end{aligned}
$$

$\Rightarrow$ changes the value of the renormalized one-loop correction

$$
\begin{aligned}
\hat{\Pi}_\gamma^{[2]}(q^2, \mu^2) &= \Pi_\gamma^{[2]}(q^2) - \Pi_\gamma^{[2]}(\mu^2) = \frac{8g^2}{(4\pi)^{D/2}} \int_0^1 dx\, x(1-x) \ln[\tfrac{m^2 + x(1-x)Q^2}{m^2 + x(1-x)\mu^2}] \\
&= \Pi_\gamma^{[2]}(q^2) - \Pi_\gamma^{[2]}(0) - (\Pi_\gamma^{[2]}(\mu^2) - \Pi_\gamma^{[2]}(0)) \\
&\approx \frac{\alpha}{3\pi} \left(\ln[\tfrac{Q^2}{m^2}] - \tfrac{5}{3} + \mathcal{O}(\tfrac{m^2}{Q^2}) - (\ln[\tfrac{\mu^2}{m^2}] - \tfrac{5}{3} + \mathcal{O}(\tfrac{m^2}{\mu^2})) \right) = \frac{\alpha}{3\pi} \ln[\tfrac{Q^2}{\mu^2}]
\end{aligned}
$$

- choosing the scale of our definitions (renormalization conditions)
  - we can change the size of the corrections relative to that scale
  - but then we have to accept different parameter values:

    $\Rightarrow$ the coupling is now defined at a different scale

Changing the scales of measurements

the renormalized coupling can be related to the bare coupling by

$$g(0) = g_0[1 + \Pi_\gamma^{[2]}(0)]^{1/2} = g_0(Z_3^{[2]}(0))^{1/2}$$

- changing the scale from 0 to μ we have

$$\begin{aligned} g(\mu) &= g_0[1 + \Pi_\gamma^{[2]}(\mu^2)]^{1/2} = g_0(Z_3^{[2]}(\mu^2))^{1/2} \\ \Rightarrow \quad g(\mu) &= g(0)[\tfrac{Z_3^{[2]}(0)}{Z_3^{[2]}(\mu^2)}]^{1/2} \approx g(0)[\tfrac{1+\Pi_\gamma^{[2]}(\mu^2)}{1+\Pi_\gamma^{[2]}(0)}]^{1/2} \approx g(0)(1 + \tfrac{1}{2}\Pi_\gamma^{[2]}(\mu^2) - \tfrac{1}{2}\Pi_\gamma^{[2]}(0)) \\ &= g(0)(1 + \tfrac{1}{2}\hat{\Pi}_\gamma^{[2]}(\mu^2)) \approx g(0)(1 + \tfrac{\alpha}{6\pi}\ln[\tfrac{\mu^2}{Am^2}]) \end{aligned}$$

- changing the scale μ' continuously from m to μ
  - the coupling is a continuous function of μ, μ', and m
  - ⇒ we can write

$$g' = g(\mu') = G(g(\mu); \tfrac{\mu'}{\mu}, \tfrac{m}{\mu})$$

- differentiating logarithmically with respect to μ'

$$\mu'\tfrac{d}{d\mu'}g' = \mu'\tfrac{d}{d\mu'}G(g(\mu); \tfrac{\mu'}{\mu}, \tfrac{m}{\mu}) = z\tfrac{\partial}{\partial z}G(g(\mu); z, \tfrac{m}{\mu})$$

- and letting μ' go to μ we get the Callan Symanzik equation

$$\mu'\tfrac{d}{d\mu'}g' \to \mu\tfrac{d}{d\mu}g = \left[\tfrac{\partial}{\partial z}G(\alpha; z, \tfrac{m}{\mu})\right]_{z=1} := \beta(g, \tfrac{m}{\mu}) \stackrel{m\ll\mu}{=} \beta(g, 0) = \beta(g)$$

  - this beta function describes the change of the coupling with the scale

Changing the scales of measurements

- the beta function integrates from one scale to the other:

$$\frac{dg}{\beta(g)} = \frac{d\mu}{\mu} = d\ln\mu \quad \Rightarrow \quad \ln\frac{\mu_2}{\mu_1} = \int_{\mu_1}^{\mu_2} \frac{dg}{\beta(g)}$$

- dimensionless quantities can always be expressed by dimensionless combinations of dimensionful quantities
  - ⇒ the ratio between the one-loop and the tree-level cross section $S = \frac{\sigma^{\text{loop}}}{\sigma^{\text{tree}}}$ can only depend on g and on ratios of scales $\frac{q^2}{\mu^2}, \frac{m}{\mu}, \ldots$
- Physics should not depend on the renormalization scale μ
  - ⇒ the logarithmic derivative of S with respect to μ should vanish:

$$\begin{aligned} 0 &= \mu\tfrac{d}{d\mu} S[g(\mu), \tfrac{q^2}{\mu^2}, \tfrac{m^2}{\mu^2}] = \left(\mu\tfrac{\partial}{\partial\mu}\Big|_{g(\mu)} + \mu\tfrac{\partial g(\mu)}{\partial\mu}\Big|_g \tfrac{\partial}{\partial g(\mu)}\Big|_\mu\right) S[g(\mu), \tfrac{q^2}{\mu^2}, \tfrac{m^2}{\mu^2}] \\ &= \left(\mu\tfrac{\partial}{\partial\mu} + \beta(g)\tfrac{\partial}{\partial g}\right) S[g, \tfrac{q^2}{\mu^2}, \tfrac{m^2}{\mu^2}] \quad \ldots \text{ Renormalization group equation for } S \end{aligned}$$

  - the normal form of the Renormalization group equation uses $\alpha = \frac{g^2}{4\pi}$ instead of g and μ^2 instead of μ:

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta(\alpha)\frac{\partial}{\partial\alpha}\right) S[\alpha, \tfrac{q^2}{\mu^2}, \tfrac{m^2}{\mu^2}] = 0$$

Changing the scales of measurements

- the Renormalization group equation for S can be solved exactly
  - by introducing the energy dependent coupling $\alpha(|q^2|)$
    - implicitly defined by

$$\ln \frac{|q^2|}{\mu^2} = \int_{\alpha(\mu)}^{\alpha(|q^2|)} \frac{d\alpha}{\beta(\alpha)}$$

    - one has to integrate and solve for the upper boundary for $\alpha(|q^2|)$
    - for simplicity we also ignored the dependence on masses, since we assumed $|q^2| \gg m^2$

$\Rightarrow$ $S = \frac{\sigma^{\text{loop}}}{\sigma^{\text{tree}}}$ **depends on $|q^2|$ only through $\alpha(|q^2|)$!**

- allows predictions to higher/lower energy scales
- gives a tool for the control of quantum corrections:
  - check of the stability of the theoretical calculation

Quantum Chromodynamics (QCD)

the bare Lagrangian including gauge-fixing

$$\mathcal{L}_0 = \bar\psi_0(i\not{D}_0 - m_0)\psi_0 - \tfrac{1}{4}G^b_{0\mu\nu}G_0^{b\mu\nu} - \tfrac{1}{2\xi_0}(\partial^\mu A^b_{0\mu})(\partial^\nu A^b_{0\nu}) + (\partial^\mu b^a)(D^{ab}_\mu c^b)$$

- quarks ψ are in the fundamental representation of $SU(3)_c$
- gluons A^b_μ are in the adjoint representation of $SU(3)_c$: $A_\mu = A^b_\mu T^b$
  - $T^b = (\frac{1}{2}\lambda^b)$ are the 8 generators of $SU(3)_c$
  - with the structure constants $[T^a, T^b] = if^{abc}T^c$
  - and the normalization $\mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}$
- the covariant derivative $D_\mu = \partial_\mu + ig_s A_\mu$ is gauge group valued
- D_μ acting on quarks: $D^{rs}_\mu \psi^s = [\delta^{rs}\partial_\mu + ig_s A^b_\mu(\frac{1}{2}\lambda^b)^{rs}]\psi^s$
  - r and s are color indices of the quarks: $r, s = 1\ldots 3$
- D_μ defines the field strength $G_{\mu\nu} = \frac{1}{ig_s}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_s[A_\mu, A_\nu]$
  - so $G^b_{\mu\nu} = 2\mathrm{Tr}[T^b G_{\mu\nu}] = \partial_\mu A^b_\nu - \partial_\nu A^b_\mu + ig_s i f^{cdb} A^c_\mu A^d_\nu = \partial_\mu A^b_\nu - \partial_\nu A^b_\mu - ig_s f^{bcd} A^c_\mu A^d_\nu$
- ghost c^a and antighost b^a are in the same representation as the gluons
  - b^a is not the anti-particle to c^a,
    - but they belong to each other like variable and conjugated momentum
  - D^{ab}_μ acting on the ghost: $D^{ab}_\mu c^b = [\delta^{ab}\partial_\mu + g_s f^{cab} A^c_\mu]c^b$

QCD in renormalized perturbation theory

- the renormalized fields $\psi = Z_2^{-1/2}\psi_0$ and $A^\mu = Z_3^{-1/2}A_0^\mu$
  - are like in QED

- the counter terms are like in QED
  - since QCD is also a vector-like theory

- Feynman rules describing the incoming and outgoing states:
  - fermion spinors contain additionally a color index
  - gauge boson polarization vectors contain additionally a group index
  - the ghosts are anticommuting scalars, but cannot appear as asymptotic states

- diagrammatic Feynman rules can be obtained from the path integral

$$Z[\eta,\bar{\eta},J^\mu;g] = N \times \int \mathcal{D}\bar{\psi}\ \mathcal{D}\psi\ \mathcal{D}b\ \mathcal{D}c\ \mathcal{D}A_\mu\ e^{i\int_x \mathcal{L}[\bar{\psi},\psi,A_\mu,b,c;g]+\bar{\psi}\eta+\bar{\eta}\psi+J^\mu A_\mu}$$

  - with the same sources as in QED
  - the Faddeev-Popov determinant $\Delta_g[A_\mu]$
    - is expressed by the path integral over the introduced ghosts b and c

QCD in renormalized perturbation theory

- Feynman rules obtained from the path integral are similar to QED
  - they have to account for the additional color structure
    - but the color structure can be treated separately: "color factor"
  - the quark propagator carries also the color of the quark: $S_F(p) = \frac{i\delta^{rs}}{\not{p}-m}$
  - the gluon propagator carries the group index: $\Delta^{ab}_{\mu\nu}(k) = \frac{i\delta^{ab}}{k^2+i\epsilon}\left(-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$
  - the quark-quark-gluon vertex includes also the group generator
    $-ig_s\Gamma^b_\mu(p,p';k) = -ig_s(2\pi)^4\delta^4(p+k-p')(\gamma_\mu)(\frac{1}{2}\lambda^b)^{rs}$
- additional Feynman rules are
  - the three- and four-gluon vertices from the $SU(3)$ self interactions
    - $V^{abc}_{\mu\nu\rho}(p,q,r) = -ig_s f^{abc}[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu}]$
    - $V^{abcd}_{\mu\nu\rho\sigma}(p,q,r,s) = -g_s^2[f^{abe}f^{ecd}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + f^{ace}f^{edb}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu}) + f^{ade}f^{ebc}(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma})]$
  - a ghost-antighost propagator $\Delta^{ab}_{\mu\nu}(k) = \frac{i\delta^{ab}}{k^2+i\epsilon}$
    - connects ghost and antighost, like the fermion propagator connects ψ and $\bar\psi$
    - carries a direction as ghosts are complex scalars
  - a ghost-antighost-gluon vertex $V^{abc}_\mu(p,p';k) = g_s f^{abc} p'_\mu$
    - the momentum is the one of the outgoing antighost
    - appears only in loops and the ghosts form a closed ghost-antighost loop

QCD in renormalized perturbation theory

- QCD has only $2 + n_f$ free physical parameters and their renormalization constants:
  - the gluon field A_μ with δZ_A (or Z_3)
  - the strong coupling constant g_s with δg_s (or Z_1)
  - the n_f (number of "flavors") quark fields ψ_f with δZ_f (or Z_2)
  - the n_f quark masses m_f with δm_f
- Feynman rules for the counter terms are the same as in QED
  (for the diagrams that appear also in QED, supplemented by color indices)
  - quark field counter term $i\delta^{rs}[(Z_2 - 1)\not{p} - \delta m]$
  - gluon field counter term $i\delta^{ab}[-g^{\mu\nu}k^2 + k^\mu k^\nu](Z_3 - 1)$
  - quark-quark-gluon vertex counter term $-ig\gamma^\mu(\frac{1}{2}\lambda^b)^{rs}(Z_1 - 1)$
- additional counter terms cannot depend on new parameters
  - that is the advantage of writing for example $g_{0s} = g_s + \delta g_s$
  - the counter terms for the three- and four-gluon vertices
    - can only depend on δg_s and δZ_A
  - the counter terms for ghost propagator and ghost-antighost-gluon vertex
    - can only come in at two- or more-loop order
    - and we restricted ourselves to the discussion of one-loop

Renormalization conditions in QCD

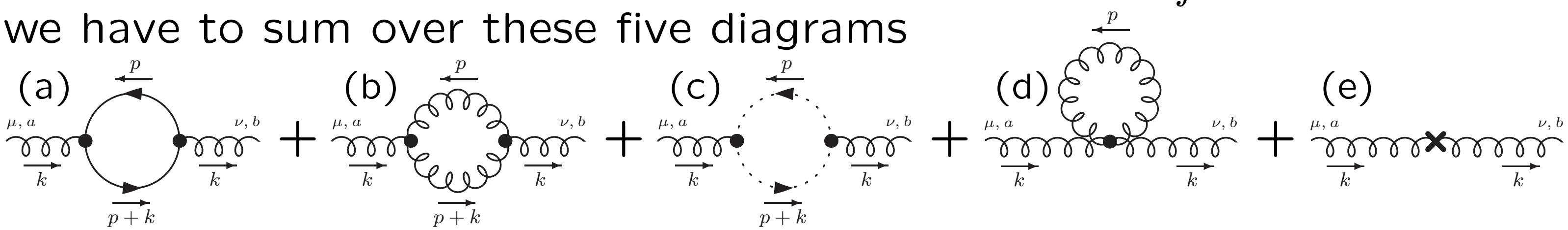
- in QED we were choosing asymptotic states
  - the full fermion propagator at $p^2 = m^2$
  - the gauge independent part of the full photon propagator at $q^2 = 0$
  - the fermion-gauge boson vertex should give the classical scattering
- in QCD we do not have these asymptotic states !
  - but we can still measure something at some scale μ
  - and relate measurements at different scales

    by the renormalization group equations (RGEs)
    - ∗ for that we have to choose gauge invariant quantities (like cross sections)
- we can still require that propagators are simple – at a certain scale!
  - that fixes the field renormalization constants δZ
  - and sets the scale, where the masses should be measured $\to \delta m$

  $\Rightarrow$ the masses are no longer defined by relating energy and momentum
    - ∗ the masses are defined as gauge invariant couplings in the Lagrangian
- the strong coupling is defined by a cross section ratio at a certain scale:
  - example: the ratio of three jet events over two jet events at the Z-pole

The gluon propagator at one-loop

- For a general result we can take $SU(N)$ with n_f quarks
  - we have to sum over these five diagrams

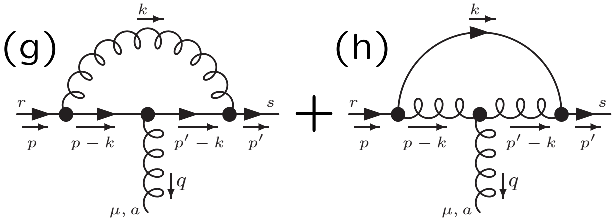


  - the sum of all of them should give a finite result
    * independent of the scale where the parameters are defined
  - we calculated (a) in QED — except for the color factor
    * which is $(T^a)^{rs}(T^b)^{sr} = \text{Tr}[T^a T^b] = C(T)\delta^{ab}$;
      for the fundamental representation N of $SU(N)$ the normalization factor is $C(N) = \frac{1}{2}$.
  - for (b), (c), and (d) the color factor is $f^{acd} f^{bdc} = C_2(G)\delta^{ab}$
    * coming from the vertices in the adjoint representation: $i f^{abc}$
  - the counter term (e) has to have the same color factor as the other diagrams
  - for the explicit calculation of (b), (c), and (d) see
    * Peskin & Schroeder, p. 522 - 526
    * Srednicki, *Quantum Field Theory*, p. 430 - 432
- the diagrams (a)-(e) determine $\delta Z_A(\mu)$ or $Z_3(\mu)$
  - in general, one has $g(\mu) = g_0 Z_1^{-1}(\mu) Z_2(\mu) Z_3^{1/2}(\mu)$; in $SU(N)$ $Z_1 \neq Z_2$

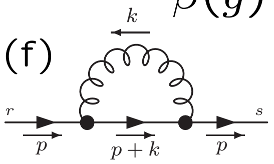
  $\Rightarrow$ for the change of $g(\mu)$, one has to calculate also Z_1 (vertex) and Z_2 (fermion field)

QCD beta function at one loop

- was the logarithmic derivative of the scale dependent coupling $g(\mu)$
  - taking $g(\mu) = g_0 Z_1^{-1}(\mu) Z_2(\mu) Z_3^{1/2}(\mu)$ we get



$$\beta(g) = \mu \frac{\partial}{\partial \mu} g_0 Z_1^{-1}(\mu) Z_2(\mu) Z_3^{1/2}(\mu)$$
$$= \frac{\mu}{g(\mu)} \left[-\frac{\partial Z_1(\mu)}{\partial \mu} + \frac{\partial Z_2(\mu)}{\partial \mu} + \frac{1}{2} \frac{\partial Z_3(\mu)}{\partial \mu} \right]$$



  - Z_2 has only a single diagram, (f), that we calculated without the color factor
    - * which is $(T^a)^{rs} \delta^{ab} (T^b)^{st} = C_2(T) \delta^{rt}$;
      for the fundamental representation N of $SU(N)$ this Casimir operator is $C_2(N) = \frac{N^2-1}{2N}$.
  - Z_1 has two diagrams, (g) and (h), as the gluon can also couple to itself
    - * the color factors are different: (g) gives $[C_2(T) - \frac{1}{2} C_2(G)] (T^b)^{rs}$;
      (h) gives $\frac{1}{2} C_2(G) (T^b)^{rs}$, but get a factor of 3 from the momentum integral
      so (g)+(h) gives $[C_2(T) + C_2(G)]$.

- taking the factors from the loops and the number of quarks n_f:

$$\beta(g) = -\frac{2g^3}{16\pi^2} \left[(C_2(T) + C_2(G)) - C_2(T) - \frac{1}{2} \left(\frac{5}{3} C_2(G) - \frac{4}{3} n_f C(N) \right) \right]$$
$$= -\frac{g^3}{16\pi^2} \left(\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(N) \right) \stackrel{SU(3)}{=} -\frac{g^3}{3(4\pi)^2} (33 - 2n_f) < 0$$

6. Quantum Field Theory (QFT) — QCD

running of the couplings

- with the beta functions we can now solve for the running of α

$$\frac{1}{2}\ln\frac{Q^2}{\mu^2} = \int_\mu^Q \frac{dg}{\beta(g)} = \int_\mu^Q \frac{4\pi\,dg}{b\,g^3} = -\frac{4\pi}{b}\left[\frac{1}{2g^2}\right]_\mu^Q = -\frac{2\pi}{b}\left[\frac{1}{g^2(Q^2)} - \frac{1}{g^2(\mu^2)}\right]$$

$$\Rightarrow \quad \alpha(Q^2) = \frac{g^2(Q^2)}{4\pi} = \frac{\frac{g^2(\mu^2)}{4\pi}}{1 - b\frac{g^2(\mu^2)}{4\pi}\ln\frac{Q^2}{\mu^2}} = \frac{\alpha(\mu^2)}{1 - b\alpha(\mu^2)\ln\frac{Q^2}{\mu^2}}$$

  - with $b_{\text{QED}} = \frac{1}{3\pi}$ and $b_{\text{QCD}} = -\frac{33-2n_f}{12\pi}$
  - one assumption in the derivation was $Q^2, \mu^2 \gg m^2$

$\Rightarrow$ with increasing energy Q^2:

  - $\alpha_{\text{em}}(Q^2)$ becomes bigger
    - * . . . and eventually hits the Landau pole at $Q = m_e e^{\frac{3\pi}{2\alpha_0}} \approx 1.2 \times 10^{277}\,\text{GeV}$
  - $\alpha_s(Q^2)$ becomes smaller $\Rightarrow$ asymptotic freedom

- with decreasing energy Q^2:
  - $\alpha_{\text{em}}(Q^2)$ becomes smaller until $\alpha_{\text{em}}(Q^2_{\text{min}} = m_e^2) = \alpha_0 = \frac{1}{137}$
  - $\alpha_s(Q^2)$ becomes larger $\Rightarrow$ confinement

6. Quantum Field Theory (QFT) — QCD

running of the couplings

- the electromagnetic coupling $\alpha_{\text{em}}(Q^2)$ is well defined for small Q^2
  - for $Q \sim 0$ we have the Thompson-limit $\alpha_{\text{em}}(0) = \alpha_0 \sim \frac{1}{137}$
  - for $Q \sim m_{\text{Plank}}$ QED is still perturbative: $\alpha_{\text{em}}(m_{\text{Plank}}) \sim \frac{1}{30}$
  - for higher energies, QFT is no longer applicable
    - ... we would need a quantum theory of gravity

- the strong coupling is measured at LEP: $\alpha_s(M_Z^2) \sim 0.1183 \pm 0.0027$
  - it shrinks to $\alpha_s(m_{\text{Plank}}) \sim 0.0022 \sim \frac{1}{440} \ll \alpha_{\text{em}}(m_{\text{Plank}})$
  - it grows to $\alpha_s(m_\tau \sim 1.776\,\text{GeV}) \sim 0.34 \pm 0.03$
  - and grows for $Q \to 1.4$ GeV $> m_{\text{Proton}}$ to infinity

- Perturbation theory cannot be used for QCD at energies $< m_\tau$
- seperating color charges uses more energy than creating a particle-antiparticle pair

  $\Rightarrow$ color charges are confined to bound states (hadrons)

$\Rightarrow$ **this is confinement**

6. Quantum Field Theory (QFT) — QCD

What do we observe from QCD?

- the bound state spectrum: baryons and mesons
  - but the energy of the bound quarks is small ... the quarks are "soft"

    ⇒ we cannot use perturbation theory
- the scattering of electrons on protons/neutrons
  - is mediated by gauge bosons (γ or Z)
  - at high momentum transfer (i.e. large recoil of the electron)
    * the gauge bosons (γ or Z) "sees" only a single quark
    * this quark gets a lot of energy ... it becomes "hard"

    ⇒ its coupling to the rest of the proton/neutron becomes small

    ⇒ the proton/neutron "fragments" ⇒ deep inelastic scattering (DIS)
- the same applies for high energy proton-(anti)proton scattering
- we can calculate the hard process using perturbative QCD (pQCD)
  - the fragmentation cannot be calculated in pQCD
  - but it is "universal": it is independent of the hard process

⇒ defining the content of the proton/neutron by

parton distribution functions (PDFs)

What we observe from QCD: parton distribution functions (PDFs)

the parton model describes the proton as an assembly of particles

- their amount depends on the energy with which one looks at the proton
- they should give the total momentum four vector of the proton
- the asymptotic description is done in the "infinite momentum frame"
  - the total momentum of the proton is much bigger than anything else
  - $\Rightarrow$ all momenta of the partons are parallel
  - $\Rightarrow$ the mass of the proton (and of the partons) can be neglected
- the i^{th} parton has a fraction x_i of the momentum of the proton P^μ: $p_i^\mu = x_i P^\mu$
  - the probability to find the parton i in the proton looking with the energy Q^2 is given by the parton distribution function $f_i(x_i, Q^2)$
- the basis set of partons for the proton has to contain quarks and the gluon
  - which quarks are taken to be in the basic set is not fixed
  - the minimal set includes gluon, up-, down-, sea-quarks, and their anti-quarks:
    - the Q^2 dependence is usually understood and not explicitely written:

$$u(x) = f_u(x_u, Q^2) \ , \quad d(x) \ , \quad s(x) \ , \quad g(x) \ , \quad \bar{u}(x) \ , \quad \bar{d}(x) \ , \quad \bar{s}(x)$$

- this basis set is universal (it does not depend on the process)

this is a similar view as we have from the QFT vacuum

What we observe from QCD: parton distribution functions (PDFs)

the parton model describes the proton as an assembly of particles

- the PDFs fulfill constraint equations:
  - a proton at rest has two up-, one down-, and no other quarks:

$$\int_0^1 dx[u(x) - \bar{u}(x)] = 2 \; , \quad \int_0^1 dx[d(x) - \bar{d}(x)] = 1 \; , \quad \int_0^1 dx[s(x) - \bar{s}(x)] = 0$$

  - the total momentum of the proton is conserved:

$$\int_0^1 dx\, x[g(x) + \sum_q (q(x) + \bar{q}(x))] = 1$$

- in DIS a parton can come from another parton by internal scattering:
  - an electron scattering on an up-quark, that afterwards emits a gluon
  - could also have scattered from an up-quark that was created by a gluon
    - we see only the electron and the "escaping" parton, anyway . . .
  - but both processes reduce the energy available in the scattering

⇒ all PDFs are connected by partial differential equations:

DGLAP equations Gribov and Lipatov (1972), Altarelli and Parisi (1977), Dokshitzer (1977)

6. Quantum Field Theory (QFT) — QCD

What we observe from QCD: parton distribution functions (PDFs)

the parton model describes the proton as an assembly of particles

- using the PDFs we can calculate hard processes in QCD
  - processes with a large energy transfer
- the soft processes are split off and "put into" the PDFs
  - this splitting is done by introducing a factorization scale μ_f:
    - processes with $Q^2 > \mu_f^2$ are calculated in pQCD
    - processes with $Q^2 < \mu_f^2$ are assumed to be described already by the PDFs
  - physics should not depend on the factorization scale

  $\Rightarrow$ similar treatment like with the RGEs
- determining the PDFs by using all available data

  is similar to setting up statistical renormalisation conditions
- the PDFs describe the proton, but only at high energies
  - what should be done at low energies? ... effective field theories (EFTs)