Quantum Electrodynamics (QED)

- the bare Lagrangian including gauge-fixing
  - bare means: one writes the Lagrangian from the theorists viewpoint, no connection to observation yet

$$\mathcal{L}_0 = \bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 - g_0\bar{\psi}_0 A\!\!\!/_0\psi_0 - \frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} - \frac{1}{2\xi_0}(\partial.A_0)^2$$

  - with the abbreviations $\partial\!\!\!/ = \gamma^\mu\partial_\mu$, $A\!\!\!/ = \gamma^\mu A_\mu$, and $(\partial.A_0) = \partial_\mu A_0^\mu$
  - and the fieldstrength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

- QED includes one charged particle (ψ) and the photon (A_μ)
  - the charged particle ψ (i.e. the electron) has the bare mass m_0
    * ψ is understood as a 4-component Dirac spinor
    * fulfilling the Dirac equation $(i\partial\!\!\!/ - m_0)\psi = 0$
    * $\bar{\psi} = \psi^\dagger\gamma^0$ is the adjoint spinor
  - and couples to the photon with the bare interactionstrength g_0
  - ξ_0 is the bare (unrenormalised) gauge-fixing parameter

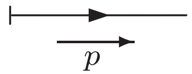
QED in renormalised perturbation theory

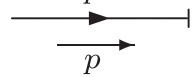
- introduces the renormalised fields $\psi = Z_2^{-1/2}\psi_0$ and $A^\mu = Z_3^{-1/2}A_0^\mu$
- the renormalised Lagrangian includes gauge-fixing

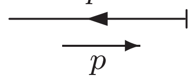
$$\begin{aligned}\mathcal{L} \;=\;& \bar\psi(i\partial\!\!\!/ - m)\psi - g\bar\psi A\!\!\!/\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2\xi}(\partial.A)^2 \\ & +\bar\psi[(Z_2-1)i\partial\!\!\!/ - \delta m]\psi - (Z_1-1)g\bar\psi A\!\!\!/\psi - \tfrac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu}\end{aligned}$$

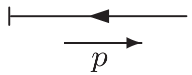
  - with field counterterms $\delta Z_\psi = Z_2 - 1$ and $\delta Z_A = Z_3 - 1$
    * since ψ is a spinor, δZ_ψ will in principle be matrix-valued, treating different helicities differently
    * but in QED the matrix is just a number times the unit matrix in spin space
  - a mass counterterm $\delta m = Z_2 m_0 - m$
  - a coupling counterterm $(Z_1 - 1) = \delta g + g(\delta Z_\psi + \frac{1}{2}\delta Z_A)$
    * the change of the coupling, δg, has to combined with the changes in the fields
  - and the redefined gauge-fixing parameter $\xi = \xi_0/Z_3$
    * in the full treatment ξ should also receive a counterterm $\delta\xi$
    * the value of $\delta\xi$ is then determined by a (new) renormalisation condition

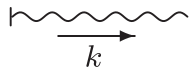
QED in renormalised perturbation theory

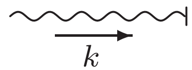
- Feynman rules describing the incoming and outgoing states
  - fermion spinors distinguish in- or outgoing particle or antiparticle
    * fermion lines carry an arrow, indicating the fermion flow
    * $u_\alpha(p,s)$ initial state particle, coming from the past 

    * $\bar{u}^\alpha(p,s)$ final state particle, going into the future 

    * $v_\alpha(p,s)$ final state antiparticle, going into the future, but fermion-arrow enters the diagram 

    * $\bar{v}^\alpha(p,s)$ initial state antiparticle, coming from the past, but fermion-arrow leaves the diagram 

    * the momentum p points into the future, s describes the helicity state
  - gauge boson polarisation vectors distinguish in- or outgoing bosons
    * in QED the gauge boson lines carry no arrow
      since there is no conserved charge connected to the photon
    * in QED the gauge boson is its own antiparticle
    * $\varepsilon^\mu(k,\lambda)$ initial state boson, coming from the past 

    * $\varepsilon^{*\mu}(k,\lambda)$ final state boson, going into the future 

    * the momentum k points into the future

QED in renormalised perturbation theory

- diagrammatic Feynman rules can be obtained from the pathintegral

$$Z[\eta,\bar\eta,J^\mu;g] = N \times \int \mathcal{D}\bar\psi\, \mathcal{D}\psi\, \mathcal{D}A_\mu\, e^{i\int_x \mathcal{L}[\bar\psi,\psi,A_\mu;g]+\bar\psi\eta+\bar\eta\psi+J^\mu A_\mu}$$

- spinors ψ and $\bar\psi$ are related by $\bar\psi = \psi^\dagger\gamma^0$
  - but used as independent variables
    - in the same way as using the complex numbers z and $\bar z = z^*$ instead of real and imaginary parts

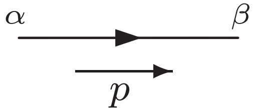
- η and $\bar\eta$ are anticommuting and spinorvalued source functions
  - ⇒ $(\bar\psi\eta)$ and $(\bar\eta\psi)$ are commuting Lorentz scalars
  - the funtional derivative $\frac{\delta}{\delta\eta}$ is also anticommuting:

$$\frac{\delta}{\delta\eta(x)}(\bar\psi(y)\eta(y)) = -\bar\psi(y)\frac{\delta}{\delta\eta(x)}\eta(y) = -\bar\psi(y)\delta(x-y) = -\bar\psi(x)$$

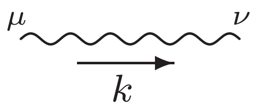
- for QED the Faddeev-Popov determinant $\Delta_g[A_\mu] = \text{Det}[\partial^2]$
  - which is constant and absorbed in the normalisation constant N
  - $\text{Det}[\partial^2]$ means summing over the spectrum of the differential operator ∂^2
    - this can be written as a pathintegral over the introduced ghosts
    - but it is completely independent from the physical fields in a $U(1)$-gauge theory

QED in renormalised perturbation theory

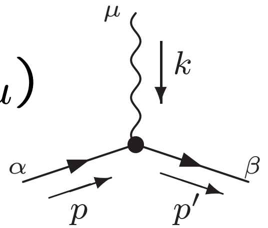
- Feynman rules obtained from the pathintegral can be pictured as
  - fermion propagator $S^{[0]}_{F\alpha}{}^{\beta}(p) = \left[\frac{i}{\not p - m}\right]_{\alpha}^{\ \beta} = i\frac{\not p_{\alpha}^{\ \beta} + m\delta_{\alpha}^{\beta}}{p^2 - m^2 + i\epsilon}$
    - ∗ carries the spinor index



    - ∗ has a direction:
      the momentum direction is counted by the propagator arrow
  - gauge boson propagator $\Delta^{[0]}_{\mu\nu}(k) = \frac{i}{k^2 + i\epsilon}\left(-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$
    - ∗ carries the vector index



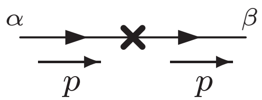
    - ∗ has no direction
      the momentum direction does not change the propagator
  - fermion-gauge boson vertex $-ig\Gamma^{[0]}_{\mu}(p, p'; k) = -ig(\gamma_\mu)$



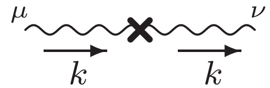
    - ∗ connects one vector index with two spinor indices
    - ∗ the momenta of the fermion follow the fermion lines,
    - ∗ the momentum of the gauge boson follows from momentum conservation
    - ∗ together with spinors and polarisation vector:
      $-ig\bar{u}(p', s')\Gamma^{[0]}_{\mu}(p, p'; k)u(p, s)\varepsilon^{(*)\mu}(k, \lambda) = -ig(2\pi)^4\delta^4(p + k - p')\bar{u}^{\alpha}(p', s')(\gamma_\mu)_{\alpha}^{\ \beta}u_{\beta}(p, s)\varepsilon^{(*)\mu}(k, \lambda)$

QED in renormalised perturbation theory

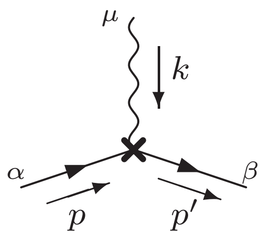
- Feynman rules for countertems can be pictured as
  - fermion field counterterm $i[(Z_2 - 1)\not{p} - \delta m]$



    - includes two renormalisation constants, Z_2 and δm
    - has two spinor indices to couple to two fermion propagators
    - has a direction:
      the momentum direction is counted by the propagator arrow
  - gauge boson field counterterm $i[-g^{\mu\nu}k^2 + k^\mu k^\nu](Z_3 - 1)$



    - includes one renormalisation constant and a projection operator
      that guarantees that the photon only couples with transverse polarisations
    - has two vector indices to couple to two gauge boson propagators
    - has no direction:
      changing of the momentum direction does not affect the counterterm
  - vertex counterterm $-ig\gamma^\mu(Z_1 - 1)$



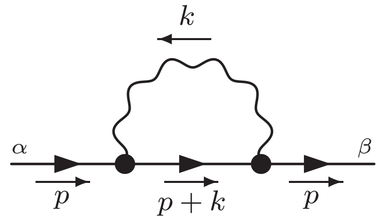
    - includes the renormalisation constants, δZ_ψ, δZ_A, and δg
    - has two spinor indices to couple to two fermion propagators
    - and a vector indices to couple to one gauge boson propagators
    - is related to the fermion field counterterm Z_2 by a Ward identity

Renormalisation conditions in QED

- somewhat similar like in the ABC-theory
- the full fermion propagator at $p^2 = m^2$ should be $S_F = \frac{i}{\not{p} - m}$
  - this fixes the mass counterterm δm
  - and the fermion field counterterm Z_2
    - by considering the fermion self energy diagram
- the gauge independent part of the full gauge boson propagator at $q^2 = 0$ should be $\Delta_{\mu\nu}(q) = \frac{-i g_{\mu\nu}}{q^2 + i\epsilon}$
  - this fixes the gauge boson field counterterm Z_3
    - by considering the gauge boson self energy diagram
- the fermion-gauge boson vertex should give the classical scattering of photons on electrons at low energies: Thomson scattering
  - this gives a condition for nearly real particles
    - the decay $e \to e + \gamma$ is kinematically not allowed
  - it also enforces the Ward identity, which gives $Z_1 = Z_2$

Elementary one-loop diagrams in QED

- are the lowest order diagrams that include loops
- can be calculated from the given Feynman rules
- fermion self energy



$$-i\Sigma^{[2]}(p) = (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S_F(p+k)\gamma^\nu \Delta_{\mu\nu}(k)$$

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu i \frac{\not{k}+\not{p}+m}{(k+p)^2-m^2+i\epsilon}\gamma^\nu \frac{i}{k^2+i\epsilon}\left(-g_{\mu\nu}+(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

- the first denominator $\mathcal{D}_1 = (k+p)^2 - m^2 = k^2 + 2k.p + p^2 - m^2$
- the numerator can be simplified using the γ-matrix identities:

$$k^{-2}\not{k}(\not{k}+\not{p}+m)\not{k} = k^{-2}(k^2\not{k} + 2(k.p)\not{k} - \not{p}k^2 + mk^2) = [\mathcal{D}_1 - (p^2-m^2)]k^{-2}\not{k} - \not{p} + m$$

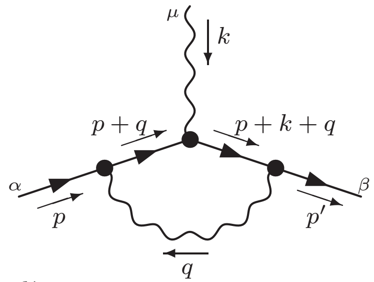
and $$-\gamma^\mu(\not{k}+\not{p}+m)\gamma_\mu = 2(\not{k}+\not{p}) - 4m$$

$$-i\Sigma^{[2]}(p) = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{2(\not{k}+\not{p})-4m}{[\mathcal{D}_1+i\epsilon][k^2+i\epsilon]} + (1-\xi)\left[\frac{\not{k}}{[k^2+i\epsilon]^2} - \frac{(p^2-m^2)\not{k}}{[\mathcal{D}_1+i\epsilon][k^2+i\epsilon]^2} - \frac{\not{p}-m}{[\mathcal{D}_1+i\epsilon][k^2+i\epsilon]}\right]$$

- the gauge dependent part vanishes for external lines:
  - the first term vanishes with the integration over d^4k
  - the second term is zero, as for external particles $p^2 = m^2$
  - the third term vanishes when acting on a spinor: $\bar{u}(p)(\not{p}-m) = (\not{p}-m)u(p) = 0$

Elementary one-loop diagrams in QED

- fermion-gauge boson vertex correction



$$
\begin{aligned}
-ig\Gamma^{[2]}_{\mu}(p,p') &= (-ig)^3 \int \frac{d^4q}{(2\pi)^4} \gamma^{\rho} S_F(p'+q)\gamma^{\mu} S_F(p+q)\gamma^{\nu}\Delta_{\rho\nu}(q) \\
&= (-ig)^3 \int \frac{d^4q}{(2\pi)^4} \gamma^{\rho} i \frac{\not{p}'+\not{q}+m}{(p'+q)^2-m^2+i\epsilon}\gamma^{\mu} i \frac{\not{p}+\not{q}+m}{(p+q)^2-m^2+i\epsilon}\gamma^{\nu} \\
&\qquad \times \frac{i}{q^2+i\epsilon}\left(-g_{\rho\nu}+(1-\xi)\frac{q_{\rho}q_{\nu}}{q^2}\right) \\
&= -g^3 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_{\nu}(\not{p}'+\not{q}+m)\gamma^{\mu}(\not{p}+\not{q}+m)\gamma^{\nu}}{[q^2+i\epsilon][(p'+q)^2-m^2+i\epsilon][(p+q)^2-m^2+i\epsilon]} \\
&\quad +(1-\xi)g^3 \int \frac{d^4q}{(2\pi)^4} \frac{\not{q}(\not{p}'+\not{q}+m)\gamma^{\mu}(\not{p}+\not{q}+m)\not{q}}{[q^2+i\epsilon]^2[(p'+q)^2-m^2+i\epsilon][(p+q)^2-m^2+i\epsilon]} = \ldots
\end{aligned}
$$

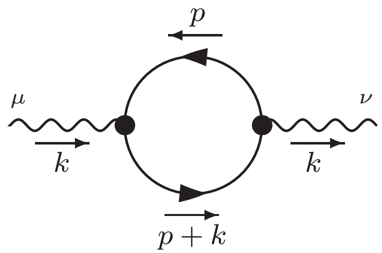
- has three denominators
- without additions it cannot describe an allowed process
  - as four momentum conservation forces the momentum of the photon to vanish
  - Thomson limit: very low energy scattering of photons on electrons $E_{\gamma} \ll m$
  - this limit is used to define the renormalisation condition :

$$
\lim_{k=p'-p\to 0} g\bar{u}(p')\Gamma^{[2]}_{\mu}(p,p')u(p) = g\bar{u}(p)\gamma_{\mu}u(p)
$$

- gives the Ward identity $Z_1 = Z_2$ from $-\frac{\partial}{\partial p^{\mu}}(\not{p}-m)^{-1} = (\not{p}-m)^{-1}\gamma_{\mu}(\not{p}-m)^{-1}$

Elementary one-loop diagrams in QED

- gauge boson self energy $i\Pi^{[2]}_{\mu\nu}(q)$



  - has an additional (-1) due to the closed fermion loop

$$
\begin{aligned}
i\Pi^{[2]}_{\mu\nu}(q) &= (-1)(-ig)^2 \int \frac{d^4p}{(2\pi)^4} (\gamma_\mu)^{\alpha}_{\delta} S_{F\alpha}{}^{\beta}(p+q)(\gamma_\nu)^{\gamma}_{\beta} S_{F\gamma}{}^{\delta}(p) \\
&= g^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_\mu i(\not p + \not q + m)\gamma_\nu i(\not p + m)]}{[(p+q)^2 - m^2 + i\epsilon][p^2 - m^2 + i\epsilon]} \\
&= -\frac{ig^2}{(4\pi)^2} \int \frac{d^4p}{i\pi^2} \frac{4[(p+q)_\mu p_\nu + p_\mu (p+q)_\nu - g_{\mu\nu}(p.(p+q) - m^2)]}{[(p+q)^2 - m^2 + i\epsilon][p^2 - m^2 + i\epsilon]}
\end{aligned}
$$

  - combining the denominators with a Feynman parameterintegral $[AB]^{-1} = \int_0^1 dx [xA + (1-x)B]^{-2}$

$$
i\Pi^{[2]}_{\mu\nu}(q) = -\frac{ig^2}{(4\pi)^2} \int_0^1 dx \int \frac{d^4p}{i\pi^2} \frac{4[(p+q)_\mu p_\nu + p_\mu (p+q)_\nu - g_{\mu\nu}(p.(p+q) - m^2)]}{[p^2 + 2xp.q + xq^2 - m^2 + i\epsilon]^2}
$$

  - replacing the integration variable $p \to p' = p + xq$ and omitting terms odd in p'
    - gives for the denominator $p^2 + 2xp.q + xq^2 - m^2 = p'^2 + x(1-x)q^2 - m^2 := p'^2 - \Delta_\gamma := \mathcal{D}$
    - and the numerator $N = (p+q)_\mu p_\nu + p_\mu (p+q)_\nu - g_{\mu\nu}(p.(p+q) - m^2)$

$$
\begin{aligned}
N &= 2p'_\mu p'_\nu + (1-2x)[q_\mu p'_\nu + p'_\mu q_\nu] - 2x(1-x)q_\mu q_\nu - g_{\mu\nu}(p'^2 + (1-2x)(p'.q) - x(1-x)q^2 - m^2) \\
&= 2p'_\mu p'_\nu - g_{\mu\nu}(p'^2 - \Delta_\gamma) + 2x(1-x)(g_{\mu\nu}q^2 - q_\mu q_\nu) + \text{ terms linear in } p'
\end{aligned}
$$

$$
\Rightarrow \quad i\Pi^{[2]}_{\mu\nu}(q) = -\frac{4ig^2}{(4\pi)^2} \int_0^1 dx \int \frac{d^4p'}{i\pi^2} \frac{2p'_\mu p'_\nu}{[\mathcal{D} + i\epsilon]^2} - \frac{g_{\mu\nu}}{[\mathcal{D} + i\epsilon]} - \frac{8ig^2}{(4\pi)^2}[g_{\mu\nu}q^2 - q_\mu q_\nu] \int_0^1 dx \int \frac{d^4p'}{i\pi^2} \frac{x(1-x)}{[\mathcal{D} + i\epsilon]^2}
$$

  - the first part vanishes when regulating the integral, the second part is transverse: $q^\mu \Pi^{[2]}_{\mu\nu}(q) = 0$

Elementary one-loop diagrams in QED

- regulating the gauge boson self energy $i\Pi^{[2]}_{\mu\nu}(q)$
  - the momentum integration d^4p averages over all possible directions
  - $\Rightarrow$ the result has to be independent from the directions of p
    - $*$ the only term possible is $g_{\mu\nu}$
  - contraction with $g^{\mu\nu}$ gives a scalar integral:

$$g^{\mu\nu}\int\frac{d^4p}{(2\pi)^4}\frac{2p_\mu p_\nu}{[\mathcal{D}+i\epsilon]^2}-\frac{g_{\mu\nu}}{[\mathcal{D}+i\epsilon]}=\int\frac{d^4p}{(2\pi)^4}\frac{2p^2}{[\mathcal{D}+i\epsilon]^2}-\frac{4}{[\mathcal{D}+i\epsilon]}$$

  - doing a Wick rotation $p^0\to ik^4$ and $\vec{p}\to\vec{k}$ we get
    - $*$ $d^4p\to id^4k$, $p^2=(p^0)^2-\vec{p}^2\to(ik^4)^2-\vec{k}^2=-k^2=:-\ell^2$, the Euclidean length
    - $*$ and we can split off the angles into a Euclidean solid angle $d\Omega^3_E$: $d^4k=\ell^3 d\ell d\Omega^3_E$
    - $*$ then we can go from our Euclidean four dimensions to D dimensions:

$$\int\frac{d^4p}{(2\pi)^4}\frac{2p^2}{[\mathcal{D}+i\epsilon]^2}-\frac{4}{[\mathcal{D}+i\epsilon]}\quad\to\quad\int\frac{i\ell^3d\ell d\Omega^3_E}{(2\pi)^4}\frac{-2\ell^2}{[-\ell^2-\Delta_\gamma+i\epsilon]^2}-\frac{4}{[-\ell^2-\Delta_\gamma+i\epsilon]}$$

$$=i\int\frac{d\Omega^3_E}{(2\pi)^4}\int_0^\infty\ell^3d\ell\frac{-2\ell^2}{[\ell^2+\Delta_\gamma]^2}+\frac{4}{[\ell^2+\Delta_\gamma]}\quad\to\quad i\int\frac{d\Omega^{D-1}_E}{(2\pi)^D}\int_0^\infty\ell^{D-1}d\ell\frac{-2\ell^2}{[\ell^2+\Delta_\gamma]^2}+\frac{D}{[\ell^2+\Delta_\gamma]}$$

  - with the integration over the solid angle $\int\frac{d\Omega^{D-1}_E}{(2\pi)^D}=\frac{2\pi^{D/2}}{(2\pi)^D\Gamma(\frac{D}{2})}=\frac{2}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}$
  - and a change of variables to $y=\frac{\Delta_\gamma}{\ell^2+\Delta_\gamma}$

Elementary one-loop diagrams in QED

- regulating the gauge boson self energy $i\Pi^{[2]}_{\mu\nu}(q)$
  - we get the boundaries $y(0) = 1$ and $y(\infty) = \frac{\Delta_\gamma}{\infty + \Delta_\gamma} = 0$
  - the measure $dy = \frac{-2\ell d\ell \Delta_\gamma}{[\ell^2+\Delta_\gamma]^2}$ and the inverse function $\ell^2 = \Delta_\gamma \frac{1-y}{y}$

  ⇒ the contracted scalar integral is

$$i\frac{2}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\int_0^\infty \ell^D \frac{-2\ell d\ell}{[\ell^2+\Delta_\gamma]^2}\left(1 - \frac{D[\ell^2+\Delta_\gamma]}{2\ell^2}\right) = i\frac{2}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\int_1^0 (\Delta_\gamma \frac{1-y}{y})^{D/2} dy \left(1 - \frac{D}{2}\frac{1}{1-y}\right)$$

$$= \quad -i\frac{2\Delta_\gamma^{D/2}}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\int_0^1 dy(1-y)^{D/2}y^{-D/2} - \tfrac{D}{2}(1-y)^{D/2-1}y^{-D/2}$$

  - recognising the definition of the Beta-function

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 dy\, y^{\alpha-1}(1-y)^{\beta-1}$$

  ⇒ the contracted scalar integral becomes (using $\Gamma(2) = 1! = \Gamma(1) = 0! = 1$)

$$-i\frac{2\Delta_\gamma^{D/2}}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\left(\frac{\Gamma(1-\frac{D}{2})\Gamma(1+\frac{D}{2})}{\Gamma(2)} - \frac{D}{2}\frac{\Gamma(1-\frac{D}{2})\Gamma(\frac{D}{2})}{\Gamma(1)}\right) = -i\frac{2\Delta_\gamma^{D/2}\Gamma(1-\frac{D}{2})}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\left(\Gamma(1+\frac{D}{2}) - \frac{D}{2}\Gamma(\frac{D}{2})\right)$$

  - since $\Gamma(z+1) = z\Gamma(z)$, the contracted scalar integral vanishes identically
- this was dimensional regularisation

Elementary one-loop diagrams in QED

- the regulated gauge boson self energy

$$i\Pi^{[2]}_{\mu\nu}(q) \;=\; -\frac{8ig^2}{(4\pi)^2}[g_{\mu\nu}q^2 - q_\mu q_\nu]\int_0^1 dx \int \frac{d^4p}{i\pi^2}\frac{x(1-x)}{[\mathcal{D}+i\epsilon]^2} =: i[g_{\mu\nu}q^2 - q_\mu q_\nu]\Pi^{[2]}_\gamma(q)$$

  has the same structure as the gauge boson field counterterm

  – together they form the renormalised one-loop selfenergy

$$i\bar{\Pi}^{[2]}_{\mu\nu}(q) = i\Pi^{[2]}_{\mu\nu}(q) - i[g_{\mu\nu}q^2 - q_\mu q_\nu](Z_3 - 1) = i[g_{\mu\nu}q^2 - q_\mu q_\nu]\bar{\Pi}^{[2]}_\gamma(q)$$

  – this can be used to resum the gauge boson propagator:

$$i\Delta_{\mu\nu} \;=\; i\Delta^{[0]}_{\mu\nu} + i\Delta^{[0]}_{\mu\rho}i\bar{\Pi}^{[2]\rho\sigma}i\Delta^{[0]}_{\sigma\nu} + i\Delta^{[0]}_{\mu\rho}i\bar{\Pi}^{[2]\rho\sigma}i\Delta^{[0]}_{\sigma\kappa}i\bar{\Pi}^{[2]\kappa\lambda}i\Delta^{[0]}_{\lambda\nu} + \ldots$$

  – since $q^\mu\bar{\Pi}^{[2]}_{\mu\nu}(q) = 0$, the gauge dependent part of $\Delta^{[0]}_{\mu\nu}$ does not contribute

    * the product $i\bar{\Pi}^{[2]\rho\sigma}i\Delta^{[0]}_{\sigma\nu} = i[g^{\rho\sigma}q^2 - q^\rho q^\sigma]\bar{\Pi}^{[2]}_\gamma(q)\frac{-ig_{\sigma\nu}}{q^2} =: P^\rho_\nu\bar{\Pi}^{[2]}_\gamma(q)$

    * where $P^\rho_\nu := [\delta^\rho_\nu - \frac{q^\rho q_\nu}{q^2}]$ is a projection operator: $P^\rho_\kappa P^\kappa_\nu = P^\rho_\nu$ and $q_\rho P^\rho_\nu = P^\rho_\nu q^\nu = 0$

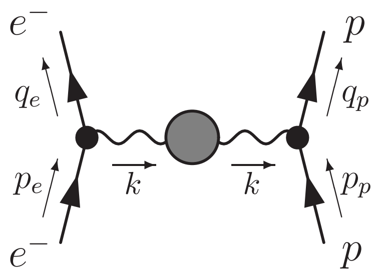
$$\Rightarrow \quad i\Delta_{\mu\nu} \;=\; i\Delta^{[0]}_{\mu\rho}(\delta^\rho_\nu + P^\rho_\nu\bar{\Pi}^{[2]}_\gamma(q) + P^\rho_\nu[\bar{\Pi}^{[2]}_\gamma(q)]^2 + \ldots) = \frac{i}{q^2}(-g_{\mu\rho} + (1-\xi)\frac{q_\mu q_\rho}{q^2})\left(\frac{P^\rho_\nu}{1-\bar{\Pi}^{[2]}_\gamma(q)} + \frac{q^\rho q_\nu}{q^2}\right)$$

$$= \frac{-iP_{\mu\nu}}{q^2[1-\bar{\Pi}^{[2]}_\gamma(q)]} - \frac{i\xi q_\mu q_\nu}{q^4} = \frac{-ig_{\mu\nu}}{q^2[1-\bar{\Pi}^{[2]}_\gamma(q)]} + \frac{iq_\mu q_\nu}{q^4}\left(\frac{1}{1-\bar{\Pi}^{[2]}_\gamma(q)} - \xi\right)$$

  – when this propagator attaches to a physical fermion line, the last term vanishes

Physics of the renormalised gauge boson propagator

- since the photon propagator describes the interaction of charges
  - one should be able to obtain the potential of a bound state
  - for that we have to consider elastic scattering
    - the particles that are bound should stay the same ...
  - and compare the QFT amplitude
    - that we calculate
  - with the QM amplitude
    - that we assume: i.e. the potential in the Schrödinger equation



- the amplitude for elastic e^-p scattering: $i(2\pi)^4\delta^4(p_e+p_p-q_e-q_p)\mathcal{M}^{\text{QFT}}$

$$= \int \frac{d^4k}{(2\pi)^4}(-igQ_e)(2\pi)^4\delta^4(p_e+k-q_e)\bar{u}(q_e)\gamma^\mu u(p_e)i\Delta_{\mu\nu}(k)(-igQ_p)(2\pi)^4\delta^4(p_p-k-q_p)\bar{u}(q_p)\gamma^\nu u(p_p)$$

$$= i(2\pi)^4\delta^4(p_e+p_p-q_e-q_p)g^2Q_eQ_p\bar{u}(q_e)\gamma^\mu u(p_e)\frac{-ig_{\mu\nu}}{k^2[1-\bar{\Pi}_\gamma^{[2]}(k)]}\bar{u}(q_p)\gamma^\nu u(p_p)$$

  - since e^- and p are on-shell , their energies do not change
  - $\Rightarrow$ $k^\mu = q_e^\mu - p_e^\mu = (0, \vec{q}_e - \vec{p}_e)$ is space-like
- the QM amplitude is $i2\pi\delta(E_e+E_p-E_e'-E_p')\mathcal{M}^{\text{QM}} = \langle\vec{q}_e,\vec{q}_p|V(\vec{k};\vec{p}_e,\vec{p}_p)|\vec{p}_e,\vec{p}_p\rangle$
  - with properly normalized wave functions for e^- and p
  - the potential should not depend on the initial momenta
    - $\Rightarrow$ $V(\vec{k};\vec{p}_e,\vec{p}_p) = V(\vec{k}) = \frac{ig_{\mu\nu}}{\vec{k}^2[1-\bar{\Pi}_\gamma^{[2]}(\vec{k})]} \approx \frac{ig_{\mu\nu}}{\vec{k}^2}[1+\bar{\Pi}_\gamma^{[2]}(\vec{k})]$

Physics of the renormalised gauge boson propagator

- evaluating the regularised gauge boson self energy

$$i\Pi_\gamma^{[2]}(q) \;=\; -\frac{8ig^2}{(4\pi)^2}\int_0^1 dx \int \frac{d^4p}{i\pi^2}\frac{x(1-x)}{[p^2+x(1-x)q^2-m^2+i\epsilon]^2}$$

  - for small momentum transfer $\vec{q}^{\,2} \ll m^2$
  - with a Wick rotation and dimensional regularisation we get

$$i\Pi_\gamma^{[2]}(q) \;=\; \frac{8ig^2}{(4\pi)^{D/2}}\int_0^1 dx\, x(1-x)(m^2+x(1-x)\vec{q}^{\,2})^{D/2-2}\Gamma(2-\tfrac{D}{2})$$

  - for taking the limit $D \to 4$ we have to expand
    * $\Gamma(\epsilon) \approx \frac{1}{\epsilon} + \gamma_{\text{E.M.}} + \ldots$
    * $(m^2+x(1-x)\vec{q}^{\,2})^\epsilon = e^{\epsilon \ln[m^2+x(1-x)\vec{q}^{\,2}]} \approx 1 + \epsilon \ln[m^2+x(1-x)\vec{q}^{\,2}] + \ldots$

$$\begin{aligned} i\Pi_\gamma^{[2]}(q) &= \frac{8ig^2}{(4\pi)^{D/2}}\int_0^1 dx\, x(1-x)(\tfrac{2}{4-D}+\gamma_{\text{E.M.}}+\ldots)(1+\tfrac{4-D}{2}\ln[m^2+x(1-x)\vec{q}^{\,2}]+\ldots) \\ &= \frac{8ig^2}{(4\pi)^{D/2}}\int_0^1 dx\, x(1-x)(\tfrac{2}{4-D}+\gamma_{\text{E.M.}}+\ln[m^2]+\ln[1+x(1-x)\tfrac{\vec{q}^{\,2}}{m^2}]+\ldots) \\ &\approx \text{const} + \frac{8ig^2}{(4\pi)^2}\int_0^1 dx\, x^2(1-x)^2\frac{\vec{q}^{\,2}}{m^2} = \text{const} + \frac{i\alpha}{15\pi}\frac{\vec{q}^{\,2}}{m^2} \end{aligned}$$

Physics of the renormalised gauge boson propagator

- Fourier transforming the potential $V(\vec{k})$
  - the renormalised gauge boson self energy has to vanish for $q^2 \to 0$

    $\Rightarrow$ $\bar{\Pi}_\gamma^{[2]}(q) = \Pi_\gamma^{[2]}(q) - \Pi_\gamma^{[2]}(0) \approx \frac{i\alpha}{15\pi}\frac{\vec{q}^2}{m^2}$
  - so $V(\vec{q}) \approx \frac{1}{\vec{q}^2}[1 + \frac{\alpha}{15\pi}\frac{\vec{q}^2}{m^2}] = \frac{1}{\vec{q}^2} + \frac{\alpha}{15\pi m^2}$
  - which gives a Fourier transformed potential

$$V(r) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}.\vec{r}} V(\vec{q}) \approx -\frac{\alpha}{r} - \frac{4\alpha^2}{15\pi m^2}\delta^3(r)$$

  $\Rightarrow$ gives part of the Lamb shift

- discussing the limit $|q^2| \gg m^2$

  $\Rightarrow$ change of the coupling strength with energy

    $\Rightarrow$ running coupling constant