Electromagnetism as a gauge theory

- classically electromagnetism (EM) was described with fields $\vec{E}$ and $\vec{B}$
- SR unified them to $F_{\mu\nu}$ with $E_i = -F_{0i}$ and $B_i = -\frac{1}{2}e_{ijk}F_{jk}$
  - scalar Φ and vector potential $\vec{A}$ to the fourvector $A_\mu = (\Phi, -\vec{A})$
  - with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- any change of the type $A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha$ does not change anything

  $\Rightarrow$ gauge transformation (reparametrisation invariance)

a gauge transformation describes the redundant parametrisation of a system

but

nature is described most exactly by gauge theories

Connection to the last semester

- In General Relativity (GR) we had
  - the invariance under changes of the coordinate systems
  - "forms" to write vectors in a basis independent way
  - the exterior derivative as a way to formulate differentiation

- using forms we could write the fieldstrength tensor from EM as a two-form F

$$\begin{aligned} F &= \tfrac{1}{2}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu F_{\mu\nu} = \tfrac{1}{2}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= \tfrac{1}{2}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \partial_\mu A_\nu + \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \partial_\nu A_\mu) \\ &= (\mathrm{d}x^\mu \partial_\mu) \wedge (\mathrm{d}x^\nu A_\nu) = \mathrm{d} \wedge A = \mathrm{d}A \ , \end{aligned}$$

  the exterior derivative of the one-form potential $A = \mathrm{d}x^\nu A_\nu$

- since the exterior derivative is nilpotent: $\mathrm{d}^2 X = \mathrm{d}(\mathrm{d}X) = 0$
  - A and $A' = A + \mathrm{d}X$ give the same F
  - $\Rightarrow$ A and A' are equivalent, related by a gauge transformation

- gauge transformations constitute local symmetries of the system

Making global symmetries local

- The simplest symmetry is just the rotation of a phase.
  - fields have to be complex to make a phase meaningful.
  - the normal Dirac spinor, describing an electron, is complex.

- Since the Lagrangian should be real, we have

$$\mathcal{L}_\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \quad \text{or} \quad \mathcal{L}_\phi = (\partial^\mu\phi^\dagger)(\partial_\mu\phi) - m^2\phi^\dagger\phi - \tfrac{1}{4!}\lambda(\phi^\dagger\phi)^2$$

- a global phase transformation $\psi \to \psi' = e^{i\alpha}\psi$ or $\phi \to \phi' = e^{i\alpha}\phi$ leaves $\mathcal{L}'_\psi = \mathcal{L}_\psi$ and $\mathcal{L}'_\phi = \mathcal{L}_\phi$ invariant.

- a local phase transformation $\psi \to \psi' = e^{i\alpha(x)}\psi$ or $\phi \to \phi' = e^{i\alpha(x)}\phi$ gives

$$\delta\mathcal{L}_\psi = \mathcal{L}'_\psi - \mathcal{L}_\psi = \bar{\psi}\gamma^\mu\psi(\partial_\mu\alpha)$$

and

$$\delta\mathcal{L}_\phi = \mathcal{L}'_\phi - \mathcal{L}_\phi = i[\phi^\dagger(\partial^\mu\phi) - (\partial^\mu\phi^\dagger)\phi](\partial_\mu\alpha) + \phi^\dagger\phi(\partial^\mu\alpha)(\partial_\mu\alpha)$$

which reminds of the conserved Noether-currents.

Making global symmetries local

- only the derivatives spoil the invariance
- changing the normal derivative to a covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu + igA_\mu$$

- so that $D_\mu\psi$ ($D_\mu\phi$) has the same transformation as ψ (ϕ):

$$D_\mu\psi \to (D_\mu\psi)' = D'_\mu\psi' = e^{i\alpha}D_\mu\psi$$

requires also a local change of the gauge field (”connection”)

$$A'_\mu = A_\mu - \frac{1}{g}(\partial_\mu\alpha)$$

- the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{1}{ig}[D_\mu, D_\nu]$
  is obviously also covariant . . . (using $D'_\mu = e^{i\alpha}D_\mu e^{-i\alpha}$):

$$F'_{\mu\nu} = \frac{1}{ig}[D'_\mu D'_\nu - D'_\nu D'_\mu] = \frac{1}{ig}[e^{i\alpha}D_\mu e^{-i\alpha}e^{i\alpha}D_\nu e^{-i\alpha} - e^{i\alpha}D_\nu e^{-i\alpha}e^{i\alpha}D_\mu e^{-i\alpha}] = e^{i\alpha}F_{\mu\nu}e^{-i\alpha}$$

and so

$$F_{\mu\nu}\psi \to (F_{\mu\nu}\psi)' = e^{i\alpha}F_{\mu\nu}e^{-i\alpha}e^{i\alpha}\psi = e^{i\alpha}F_{\mu\nu}\psi$$

Making global symmetries local

- The Lagrangians of QED

$$\mathcal{L}_\psi = i\bar{\psi}\gamma^\mu D_\mu \psi - \tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - m\bar{\psi}\psi$$

or scalar QED

$$\mathcal{L}_\phi = (D^\mu \phi^\dagger)(D_\mu \phi) - \tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - m^2 \phi^\dagger \phi - \tfrac{1}{4}\lambda(\phi^\dagger \phi)^2$$

are invariant under the gauge transformations

$$\psi \overset{\alpha}{\to} \psi' = e^{i\alpha}\psi \qquad \phi \overset{\alpha}{\to} \phi' = e^{i\alpha}\phi \qquad A_\mu \overset{\alpha}{\to} A'_\mu = A_\mu - \tfrac{1}{g}(\partial_\mu \alpha)$$

- The gauge symmetry forbids any mass term for the gauge field A_μ !

$$m_A^2 A^2 = m_A^2 A^\mu A_\mu \overset{\alpha}{\to} m_A'^2 A'^\mu A'_\mu = m_A'^2 (A^2 - \tfrac{2}{g} A^\mu (\partial_\mu \alpha) + \tfrac{1}{g^2}(\partial^\mu \alpha)(\partial_\mu \alpha))$$

for arbitrary α this can only be invariant if

$$m_A^2 = m_A'^2 = 0$$

The gauge boson propagator

- Canonical quantisation (CQ) defines the propagator $\Delta_{\mu\nu}(x-y)$
  - as the time-ordered product

$$\Delta_{\mu\nu}(x-y) = \langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle$$

- from the pathintegral we get

$$i\Delta_{\mu\nu}(x-y) = \frac{1}{Z(0;g)} \frac{\delta^2 Z(J;g)}{i\delta J^\mu(x)\, i\delta J^\nu(y)}\bigg|_{J=0}$$

- but both cannot be calculated !
  - in CQ the conjugate momentum of A_0 does not exist
  - in the pathintegral, one cannot invert the term bilinear in the field

$$-\tfrac{1}{4}\int_x F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}\int_x A_\nu \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \tfrac{1}{2}\int_x A_\nu(\eta^{\nu\rho}\partial_\mu\partial^\mu - \partial^\nu\partial^\rho)A_\rho$$

    - as the operator $P^{\nu\rho} = (\eta^{\nu\rho}\partial_\mu\partial^\mu - \partial^\nu\partial^\rho)$ is singular:

$$P^{\nu\rho}\partial_\rho = (\eta^{\nu\rho}\partial^2 - \partial^\nu\partial^\rho)\partial_\rho = \partial^2\partial^\nu - \partial^\nu\partial^2 = 0$$

- ... that does not happen if there would be a mass term
  - but the mass term is forbidden by the gauge symmetry ...

Gauge bosons in canonical quantisation

- Canonical quantisation uses the Hamiltonian
  - but the Legendre transform of the Lagrangian is not defined for a variable that does not have a conjugate momentum
    - as is the case for A_0 in $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$ (using $E_j = F_{0j}$ and $B_j = -\frac{1}{2}\epsilon_{jk\ell}F_{k\ell}$)
    - $\dot{A}_0 = \partial_0 A_0$ does not appear in the Lagrangian
    - the conjugate momentum for A_i is $\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}_i - \nabla_i A_0 = E_i$

  ⇒ no Legendre transform with respect to A_0 is possible
  - making the Legendre transform with the well defined pair (A_i, π^i)

$$\mathcal{H} = \pi^i \dot{A}_i - \mathcal{L} = E_i(E_i + \nabla_i A_0) - \frac{1}{2}(\vec{E}^2 - \vec{B}^2) = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \vec{E} \cdot \vec{\nabla} A_0$$

  - partial integration gives $\mathcal{H} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) - (\vec{\nabla} \cdot \vec{E})A_0$
  - A_0 acts like a Lagrange multiplier for the constraint $\vec{\nabla} \cdot \vec{E} = 0$ (Gauss law)

- the canonical transformation produces a ”first class” constraint
  - including this constraint with the Dirac bracket

Gauge bosons in canonical quantisation

- the Dirac bracket allows "normal" canonical quantisation
  - but the constraint $\vec{\nabla} \cdot \vec{E} = 0$ violates Lorentz covariance
  - when choosing $A_0 = 0$, the constraint can be written covariantly
    - Lorentz gauge: $\partial_\mu A^\mu = 0$
- quantization follows the Gupta-Bleuer procedure
  - which introduces negative norm states (ghosts)
  - these ghosts subtract the unphysical degrees of freedom
- the gauge fields are Fourier transformed
  - coefficient functions become creation and annihilation operators
  - these have to be orthogonal to the momentum (Gauss law)

  $\Rightarrow$ only two polarisation vectors $\varepsilon_\mu^{(j)}$ $\Rightarrow$ transverse photons
- the propagator becomes

$$\Delta_{\mu\nu}(x-y) = \langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle = \int \frac{d^4k}{(2\pi)^4} \sum_{j=1}^{2} \varepsilon_\mu^{(j)}(k)\varepsilon_\nu^{(j)}(k) \frac{ie^{-ik.(x-y)}}{k^2+i\epsilon}$$

Gauge bosons in the pathintegral

- in the pathintegral we also integrate over all field configuratuions

$$Z[J] = \int \mathcal{D}A_\mu \, e^{iS + iJ^\mu A_\mu} = \int \mathcal{D}A_\mu \, e^{iS[A,J]}$$

  - also the ones that are physically equivalent
    - because the differ only by gauge transformations
  - A_μ transforms under a gauge transformation $U(x)$: $A_\mu \to A_\mu^U \neq A_\mu$
    - for EM we have the gauge group $U(1)$ and $U(x) = e^{i\varphi(x)}$, so $A_\mu^U = A_\mu - \frac{1}{g}\partial_\mu \varphi$
  - but S and $\mathcal{D}A_\mu$ stay invariant under the gauge transformation

  $\Rightarrow$ $Z[J]$ picks up a divergent factor $\sim v(G)^V$

  (with $v(G)$ the volume of the gauge group and V the volume of space-time)

- Faddeev and Popov found a way of factoring out this constant factor
  - by imposing a gauge condition $g(A_\mu^U)$ consistently
  - using the Faddeev-Popov determinant $\Delta_g^{-1}[A_\mu] = \int \mathcal{D}U \delta(g(A_\mu^U))$
  - that $\Delta_g^{-1}[A_\mu] = \Delta_g^{-1}[A_\mu^U]$ is invariant under gauge transformations can be seen
    - by considering the group structure of the gauge transformations:
    - when U and U' are gauge transformations, so is $U'' = U'U$

Gauge bosons in the pathintegral

- inserting $1 = \Delta_g[A_\mu] \int \mathcal{D}U \delta(g(A_\mu^U))$ into the partition function

$$Z[J] = \int \mathcal{D}A_\mu \; \Delta_g[A_\mu] \int \mathcal{D}U \, \delta(g(A_\mu^U)) \; e^{iS[A,J]}$$

  - changing $A_\mu \to A_\mu^{U'}$ as a change of integration variables
    - with U' an arbitrary gauge transformation
  - using the gauge invariance of $S[A,J]$, $\mathcal{D}A_\mu$, and $\Delta_g[A_\mu]$ we get

$$Z[J] = \left[\int \mathcal{D}U\right] \int \mathcal{D}A_\mu \; \Delta_g[A_\mu] \; e^{iS[A,J]} \delta(g(A_\mu^{U'U}))$$

    - for $U' = U^{-1}$ nothing depends on U $\Rightarrow$ $\left[\int \mathcal{D}U\right]$ is really a constant
  - this procedure fixes the gauge consistently
    - if $\Delta_g[A_\mu]$ is finite and non vanishing identically

- changing the gauge condition to $\delta(g(A_\mu) - c(x))$
  - and averaging (integrating with gaussian weight) over the function $c(x)$ gives

$$Z[J] = \int \mathcal{D}A_\mu \, \mathcal{D}c \, e^{-i\int_x \frac{c^2}{2\xi}} \Delta_g[A_\mu] \; e^{iS[A,J]} \delta(g(A_\mu) - c(x)) = \int \mathcal{D}A_\mu \; \Delta_g[A_\mu] \; e^{i\int_x \mathcal{L}[A,J] - \frac{g(A_\mu)^2}{2\xi}}$$

$\Rightarrow$ a gauge fixed Lagrangian

Gauge bosons in the pathintegral

- the gauge fixed Lagrangian (in Lorentz gauge $\partial_\mu A^\mu = 0$)

$$\mathcal{L}_\xi = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2\xi} (\partial_\mu A^\mu)^2$$

  - allows the construction of a general gauge dependent propagator
    * as the operator $P^{\nu\rho} = (\eta^{\nu\rho}\partial^2 - (1 - \frac{1}{\xi})\partial^\nu \partial^\rho)$ is no longer singular

$$i\Delta^{g^0}_{\mu\nu}(x-y) = \frac{1}{Z(0;0)} \frac{\delta^2 Z(J;0)}{i\delta J^\mu(x)\, i\delta J^\nu(y)}\bigg|_{J=0} = [\eta^{\mu\nu}\partial^2 - (1 - \tfrac{1}{\xi})\partial^\mu \partial^\nu]^{-1}$$

- as in 2.QTF lecture, the free propagator is the inverse of the bilinear of the fields
  - from the Lagrangian we have after partial integration

$$\frac{1}{2} \int_x A_\mu (\eta^{\mu\nu}\partial_\rho \partial^\rho - \partial^\mu \partial^\nu) A_\nu + \tfrac{1}{2\xi} A_\mu (\partial^\mu \partial^\nu A_\nu)$$

  - with the Fourier transformation we can replace $\partial_\mu \to -ip_\mu$
  - this inverse can be calculated by $P^{\nu\rho}\Delta_{\rho\mu} = \delta^\nu_\mu$ and the ansatz $\Delta_{\rho\mu} = A\eta_{\rho\mu} + Bp_\rho p_\mu$

$$\delta^\nu_\mu = -[\eta^{\nu\rho}p^2 - (1 - \tfrac{1}{\xi})p^\nu p^\rho](A\eta_{\rho\mu} + Bp_\rho p_\mu) = -p^2(A\delta^\nu_\mu + Bp^\nu p_\mu) + (1 - \tfrac{1}{\xi})p^\nu p_\mu (A + Bp^2)$$

$\Rightarrow$ $A = -p^{-2}$ and $B = (1 - \frac{1}{\xi})/(\frac{1}{\xi}p^2)A = (1 - \xi)p^{-4}$ and the propagator in R_ξ gauge

$$i\Delta^{g^0}_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - i\epsilon} (-\eta_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2})$$

Gauge bosons in the pathintegral

- treating the Faddeev-Popov determinant consistently
  - introduces again ghost fields (”book-keeping fields”)
  - in $U(1)$ gauge groups this can be avoided
    - as the determinant is constant
- a general framework is given by the BRST quantisation
  - introduced in the 1970s by Becchi, Rouet, Stora and independently Tyutin
  - it introduces a nilpotent operator to deal with the gauge degrees of freedom
  - it generates a ”supersymmetry” that allows to project out the ghosts
- with the reformulation of QFT in terms of fiber bundles
  - BRST can be understood as a geometrical operation on the fiber bundle
  - enforcing an ”anomaly cancellation” of the ghost
  - ... connection to General Relativity
- for gravity and supergravity one has to generalise the formalism

  $\Rightarrow$ Batalin-Vilkovisky formalism
    - and lots and lots of more ghosts ...