Canonical Quantisation

- starting with definition of the S-matrix $S_{\beta\alpha} = \langle \Psi_\beta^+ | \Psi_\alpha^- \rangle$
- multiplying the Lippmann-Schwinger equations with $\langle \Phi_\beta | V$ we get

$$T^{\pm}_{\beta\alpha} = \langle \Phi_\beta | V | \Psi^{\pm}_\alpha \rangle = \langle \Phi_\beta | V | \Phi_\alpha \rangle + \int d\gamma \frac{\langle \Phi_\beta | V | \Phi_\gamma \rangle T^{\pm}_{\gamma\alpha}}{E_\alpha - E_\gamma \pm i\epsilon}$$

$$= V_{\beta\alpha} + \int d\gamma \frac{V_{\beta\gamma} V_{\gamma\alpha}}{E_\alpha - E_\gamma \pm i\epsilon} + \int d\gamma d\gamma' \frac{V_{\beta\gamma} V_{\gamma\gamma'} V_{\gamma'\alpha}}{(E_\alpha - E_\gamma \pm i\epsilon)(E_\alpha - E_{\gamma'} \pm i\epsilon)} + \dots$$

where $V_{\beta\alpha} = \langle \Phi_\beta | V | \Phi_\alpha \rangle$.

  – we get old fashioned perturbation theory.

- and looking at the asymptotic behavior $t \to \infty$, we get (assuming $(T^+_{\alpha\beta})^\dagger = T^-_{\beta\alpha}$)

$$S_{\beta\alpha} = \delta(\beta - \alpha) + 2i\pi\delta(E_\beta - E_\alpha) T^-_{\beta\alpha}$$

so $T^+_{\beta\alpha}$ describes the interaction.

Canonical Quantisation

- writing the $\mathbf{S}$-operator as $S_{\beta\alpha} = \langle \Phi_\beta | \mathbf{S} | \Phi_\alpha \rangle$
- we get $\mathbf{S} = \Omega(\infty)^\dagger \Omega(-\infty) = U(+\infty, -\infty)$
- where $U(\tau, \tau_0) = \Omega(\tau)^\dagger \Omega(\tau_0) = e^{iH_0\tau} e^{-iH\tau} e^{iH\tau_0} e^{-iH_0\tau_0}$
- so $i\frac{d}{d\tau} U(\tau, \tau_0) = e^{iH_0\tau} V e^{-iH_0\tau} U(\tau, \tau_0) =: V(\tau) U(\tau, \tau_0)$
- with the solution $U(\tau, \tau_0) = 1 - i \int_{\tau_0}^{\tau} dt\, V(t) U(t, \tau_0)$

$$
\begin{aligned}
&= 1 - i \int_{\tau_0}^{\tau} dt_1\, V(t_1) + (-i)^2 \int_{\tau_0}^{\tau} dt_1 \int_{\tau_0}^{t_1} dt_2\, V(t_1) V(t_2) + \ldots \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{\tau_0}^{\tau} dt_1 dt_2 \ldots dt_n\, T\{V(t_1) V(t_2) \ldots V(t_n)\} \\
&= T \exp\left(-i \int_{\tau_0}^{\tau} dt\, V(t) \right)
\end{aligned}
$$

Canonical Quantisation

- evaluating now $S_{\beta\alpha}$ with $\beta = \phi(x_4)\phi(x_3)$ and $\alpha = \phi(x_2)\phi(x_1)$

$$S_{\beta\alpha} = \langle\phi(x_3)\phi(x_4)|T \exp\left(-i\int dt\, V(t)\right)|\phi(x_1)\phi(x_2)\rangle$$

- taking $V(t) = \int_{\vec{z}} \frac{g}{4!}\phi^4(z)$ or $\int dt\, V(t) = \frac{g}{4!}\int dz\, \phi^4(z)$

- splitting $\phi = \phi^+ + \phi^-$ into the annihilation and the creation part

$$\phi^-(t,\vec{x}) = \int \frac{d^3\vec{k}\, e^{-ik.x} a(\vec{k})}{(2\pi)^{3/2}(2E_{\vec{k}})^{1/2}} \qquad \text{and} \qquad \phi^+(t,\vec{x}) = \int \frac{d^3\vec{k}\, e^{ik.x} a^\dagger(\vec{k})}{(2\pi)^{3/2}(2E_{\vec{k}})^{1/2}}$$

- since $t_1, t_2 < t < t_3, t_4$ and writing $\phi_i = \phi(x_i)$

$$\begin{aligned} S_{\beta\alpha} &= T\langle 0|\phi_3^-\phi_4^- e^{\frac{-ig}{4!}\int dz \phi_z^4}\phi_1^+\phi_2^+|0\rangle \\ &= T\langle 0|\phi_3^-\phi_4^-\left(1 - \frac{ig}{4!}\int dz\, \phi_z^4 + \frac{1}{2}(\frac{-ig}{4!})^2\int dz_1 dz_2\, \phi_{z_1}^4\phi_{z_2}^4 + \dots\right)\phi_1^+\phi_2^+|0\rangle \end{aligned}$$

Canonical Quantisation

- a normal ordered product of fields has a or ϕ^- to the right of $a^\dagger$ or ϕ^+:
  - $N\{\phi^-(x)\phi^+(y)\} := \phi^+(y)\phi^-(x)$
  - it has vanishing vacuum expectation value: $\langle 0|N\{\dots\}|0\rangle = 0$
- a time ordered product has the smaller time to the right:
  - $T\{\phi(x)\phi(y)\} := \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)$
  - it is related to the normal ordered product by Wick contractions:

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y)\} + D_F(x-y)$$

- because

$$\begin{aligned}
T\{\phi(x)\phi(y)\} &= \theta(x^0 - y^0)(\phi_x^+\phi_y^+ + \phi_x^+\phi_y^- + \phi_x^-\phi_y^+ + \phi_x^-\phi_y^-) + \theta(y^0 - x^0)(\dots) \\
&= \theta(x^0 - y^0)(\phi_x^+\phi_y^+ + \phi_x^+\phi_y^- + [\phi_x^-, \phi_y^+] + \phi_y^+\phi_x^- + \phi_x^-\phi_y^-) + \theta(y^0 - x^0)(\dots) \\
&= N\{\phi(x)\phi(y)\} + \theta(x^0 - y^0)\langle 0|[\phi_x^-, \phi_y^+]|0\rangle + \theta(y^0 - x^0)\langle 0|[\phi_y^-, \phi_x^+]|0\rangle \\
&= N\{\phi(x)\phi(y)\} + D_F(x-y)
\end{aligned}$$

Canonical Quantisation

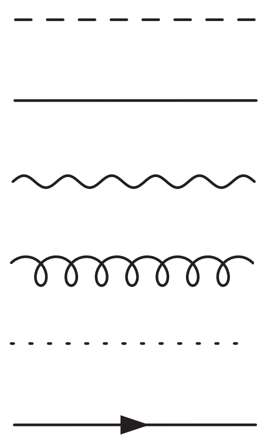
- so with $D_F(x_j - x_k) = D_{Fjk}$ we get

$$
\begin{aligned}
S_{\beta\alpha} &= \langle 0|T\{\phi_3^-\phi_4^-\phi_1^+\phi_2^+\}|0\rangle - \frac{ig}{4!}\int dz\,\langle 0|T\{\phi_3^-\phi_4^-\phi_z\phi_z\phi_z\phi_z\phi_1^+\phi_2^+\}|0\rangle + \dots \\
&= \langle 0|(N\{\phi_3^-\phi_4^-\phi_1^+\phi_2^+\} + N\{\phi_4^-\phi_1^+\}D_{F23} + N\{\phi_3^-\phi_1^+\}D_{F24} \\
&\qquad +D_{F23}D_{F14} + 1 \leftrightarrow 2)|0\rangle \\
&\quad -\frac{ig}{4!}\int dz\,\langle 0|T\{\phi_3^-\phi_4^-\phi_z\phi_z\phi_z\phi_z\phi_1^+\phi_2^+\}|0\rangle + \dots \\
&= D_{F13}D_{F24} + D_{F23}D_{F14} - ig\int dz\,D_{F1z}D_{F2z}D_{Fz3}D_{Fz4} \\
&\quad -\frac{ig}{2!}\int dz\,(D_{F13}D_{F2z}D_{Fzz}D_{Fz4} + D_{F14}D_{F2z}D_{Fzz}D_{Fz3} \\
&\qquad +D_{F13}D_{F24}D_{Fzz}D_{Fzz} + 1 \leftrightarrow 2) + \dots
\end{aligned}
$$

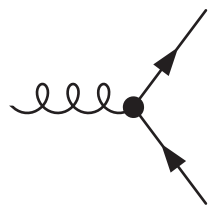
- the first two terms are just "no interaction"
- the first term with g is the scattering
- the other terms are "loop corrections" to the propagators

Diagrams

- each propagator is a line
  - for scalars — a dashed line
  - for fermions — a straight line
  - for photon, W-, and Z-boson — a wavy line
  - for gluons — a springy line
  - for ghosts — a dotted line
  - if a charge flows, the line has an arrow



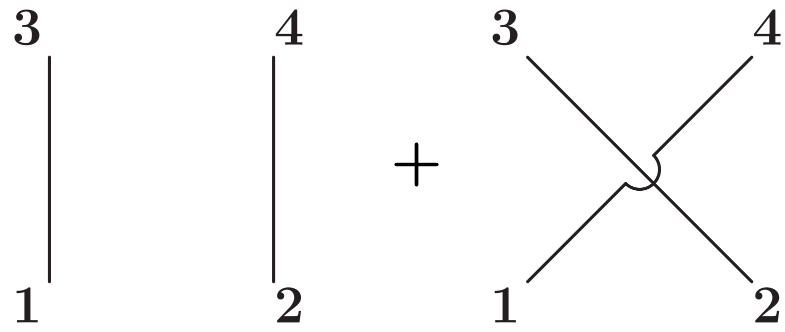
- for the ϕ^4-theory, the propagator is a simple line
- each coupling is a dot
  - connecting the lines

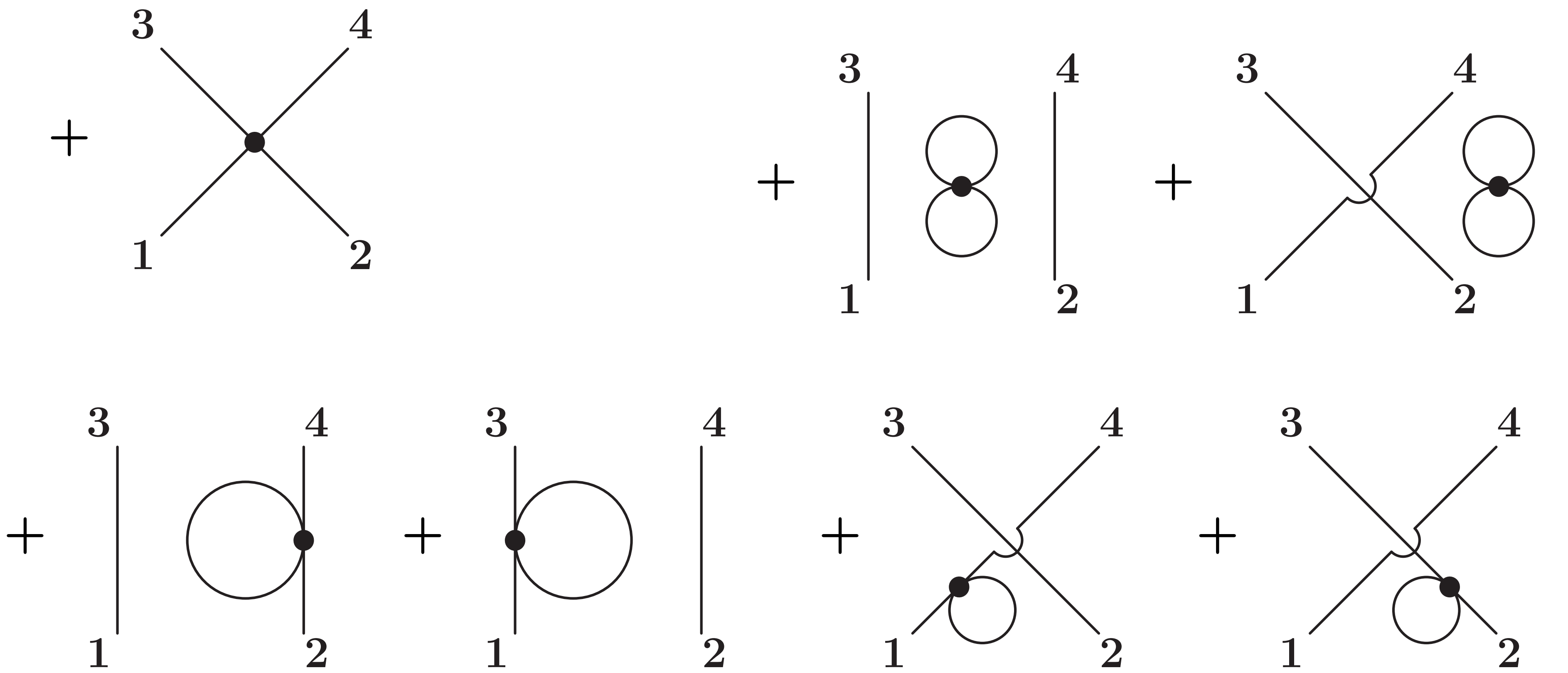


- the diagram can flow from bottom up — like SR worldlines
- the diagram can flow from left to right — like writing

Diagrams

- $S_{\beta\alpha}$ is then represented as





Path Integral Quantisation

$$Z(J) = \int \mathcal{D}\phi \, e^{i(S+\int_x J\phi)} \qquad \text{with} \qquad S = -\int d^4x \, \tfrac{1}{2}\phi(\partial^2 + m^2)\phi + \tfrac{g}{4!}\phi^4$$

- splitting S into the potential and the bilinear (= kinetic) part
- writing the kinetic part as $-\frac{1}{2}\phi(\partial^2 + m^2)\phi = \frac{1}{2}\phi D_F^{-1}\phi$

$$Z(J;g) = \int \mathcal{D}\phi \;\; e^{i\int \frac{1}{2}\phi D_F^{-1}\phi + J\phi} \;\; e^{-\frac{ig}{4!}\int \phi^4}$$

- expanding the potential as a power series $e^{\frac{-ig}{4!}\int \phi^4} = \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{-ig}{4!})^n [\int \phi^4]^n$
- replacing in the series $\phi \to \frac{\delta}{i\delta J}$ and putting ϕ-independent parts in front:

$$Z(J;g) = \sum_{n=0}^{\infty} \frac{1}{n!}(\frac{-ig}{4!})^n \left[\int d^4x \left(\frac{\delta}{i\delta J}\right)^4\right]^n \int \mathcal{D}\phi \;\; e^{i\int \frac{1}{2}\phi D_F^{-1}\phi + J\phi}$$

- solving the Gaussian integral and rewriting the sum as an exponent:

$$Z(J;g) = N e^{\frac{-ig}{4!}\int (\frac{\delta}{i\delta J})^4} \;\; e^{-\frac{i}{2}\int_{x,y} J(x) D_F(x-y) J(y)}$$

Path Integral Quantisation

- the free propagator is obtained with $g = 0$:

$$D_F^0(x-y) = \frac{i}{N} \frac{\delta^2 N e^{-\frac{i}{2}\int_{x,y} J(x) D_F(x-y) J(y)}}{\delta J(x)\, \delta J(y)}\Bigg|_{J=0} = \frac{i}{Z(0;0)} \frac{\delta^2 Z(J;0)}{\delta J(x)\, \delta J(y)}\Bigg|_{J=0}$$

- the full propagator is obtained with general g:

$$iD_F(x-y) := \frac{1}{Z(0;g)} \frac{\delta^2 Z(J;g)}{i\delta J(x)\, i\delta J(y)}\Bigg|_{J=0}$$

- the previous example, $S_{\beta\alpha}$, is a four-point function:

$$S_{\beta\alpha} = \frac{\int \mathcal{D}\phi \;\; \phi_1\phi_2\phi_3\phi_4 e^{i(S+\int_x J\phi)}}{\int \mathcal{D}\phi \;\; e^{i(S+\int_x J\phi)}} = \frac{1}{Z(0;g)} \frac{\delta^4 Z(J;g)}{\delta J_1\, \delta J_2\, \delta J_3\, \delta J_4}\Bigg|_{J=0}$$

- evaluating $S_{\beta\alpha}$ by expanding the exponent to the desired order of g.

Path Integral Quantisation

- to order g^1 we get $S_{\beta\alpha} = \frac{1}{Z(0;g^1)} \frac{\delta^4 Z(J;g^1)}{\delta J_1\, \delta J_2\, \delta J_3\, \delta J_4}\Big|_{J=0}$

- now $Z(J;g^1) = N \left(1 + \frac{ig}{4!} \int_z (\frac{\delta}{i\delta J_z})^4\right) e^{-\frac{i}{2}\int_{x,y} J_x D_{xy} J_y}\Big|_{J=0}$

  - all Js from the second factor have to be differentiated
  - the remaining Js are set to zero

- abbreviating $\int_{x,y} J_x D_{xy} J_y = D_{JJ}$ and $\int_y D_{zy} J_y = D_{zJ}$
- so $Z(0;g^1) = N \left(1 + \frac{ig}{4!} \int_z (\frac{\delta}{i\delta J_z})^4\right) \left(1 - \frac{i}{2} D_{JJ} + \frac{1}{2}[-\frac{i}{2} D_{JJ}]^2\right) |_{J=0}$
  - we have 0 or 4 derivatives of J, so the middle term vanishes.

$$
\begin{aligned}
Z(0;g^1) &= N \left(1 + \frac{ig}{4!} \int_z (\frac{\delta}{\delta J_z})^4 \frac{1}{8} (iD_{JJ})^2\right) \\
&= N \left(1 + \frac{ig}{8} \int_z (iD_{zz})(iD_{zz})\right)
\end{aligned}
$$

Path Integral Quantisation

$$\left.\frac{\delta^4 Z(J;g^1)}{\delta J_1\,\delta J_2\,\delta J_3\,\delta J_4}\right|_{J=0} = N\frac{\delta^4}{\delta J_1\,\delta J_2\,\delta J_3\,\delta J_4}\left(1+\frac{ig}{4!}\int_z(\frac{\delta}{\delta J_z})^4\right)e^{-\frac{i}{2}D_{JJ}}\Bigg|_{J=0}$$

- here we have 4 or 8 derivatives
- the exponent has to be expanded to these orders ($n = 2, 4$):

$$e^{-\frac{i}{2}D_{JJ}} \propto \frac{1}{2}\left[-\frac{i}{2}D_{JJ}\right]^2 + \frac{1}{4!}\left[-\frac{i}{2}D_{JJ}\right]^4 = e_2 + e_4$$

- differentiating gives:

$$\left.\frac{\delta^4 Z(J;g^1)}{\delta J_1\,\delta J_2\,\delta J_3\,\delta J_4}\right|_{J=0} = \frac{\delta^4 e_2}{\delta J_1\,\delta J_2\,\delta J_3\,\delta J_4} + \frac{ig}{4!}\int_z \frac{\delta^8 e_4}{\delta J_1\,\delta J_2\,\delta J_3\,\delta J_4\,(\delta J_z)^4}$$

where

$$\frac{\delta^4 e_2}{\delta J_1\,\delta J_2\,\delta J_3\,\delta J_4} = (iD_{12})(iD_{34}) + (iD_{13})(iD_{24}) + (iD_{14})(iD_{23}) =: D_{\text{tree}}$$

Path Integral Quantisation

- with writing $\delta_i = \frac{\delta}{\delta J_1}$ and $\delta_{1234} = \delta_1\,\delta_2\,\delta_3\,\delta_4$ we get

$$\begin{aligned}
\delta_{1234}\,(\delta_z)^4 e_4 &= \delta_{1234}\,(\delta_z)^3\,\tfrac{1}{4!}\,4[\tfrac{i}{2}D_{JJ}]^3(iD_{zJ}) \\
&= \delta_{1234}\,(\delta_z)^2\,\left(\tfrac{1}{3!}\,3[\tfrac{i}{2}D_{JJ}]^2(iD_{zJ})^2 + \tfrac{1}{3!}\,[\tfrac{i}{2}D_{JJ}]^3(iD_{zz})\right) \\
&= \delta_{1234}\,\delta_z\,\left([\tfrac{i}{2}D_{JJ}](iD_{zJ})^3 + \tfrac{3}{2}\,[\tfrac{i}{2}D_{JJ}]^2(iD_{zJ})(iD_{zz})\right) \\
&= \delta_{1234}\,\left((iD_{zJ})^4 + 6[\tfrac{i}{2}D_{JJ}](iD_{zJ})^2(iD_{zz}) + \tfrac{3}{2}\,[\tfrac{i}{2}D_{JJ}]^2(iD_{zz})^2\right)
\end{aligned}$$

- the first term immediately gives $4!\,(iD_{z1})(iD_{z2})(iD_{z3})(iD_{z4})$
- the middle term gives $12\,(iD_{ab})(iD_{zc})(iD_{zd})(iD_{zz})$
  where $abcd = \mathcal{P}(1234)$ is a permutation of 1234.
- the last term gives $D_{\text{tree}} \cdot 3\,(iD_{zz})^2$
- we have to set $D_{12} = D_{34} = 0$, as we are looking at $12 \to 34$
  - this comes from "wave-function factors" outside the Path Integral

Path Integral Quantisation

- $\delta_{1234} Z(J; g^1)\Big|_{J=0}$

$$= D_{\text{tree}} \left(1 + \frac{ig}{4!} \int_z 3\,(iD_{zz})^2\right) + \frac{ig}{4!} \int_z 4!\,(iD_{z1})(iD_{z2})(iD_{z3})(iD_{z4})$$
$$+ \frac{ig}{4!} \int_z 12\,(iD_{ab})(iD_{zc})(iD_{zd})(iD_{zz})$$

- $S_{\beta\alpha} = \delta_{1234} Z(J; g^1)\Big|_{J=0} Z(0; g^1)^{-1}$

$$= D_{\text{tree}} + ig \int_z \left[(iD_{z1})(iD_{z2})(iD_{z3})(iD_{z4}) + \tfrac{1}{2}(iD_{ab})(iD_{zc})(iD_{zd})(iD_{zz})\right]$$
$$\times \left(1 + \frac{ig}{8} \int_z (iD_{zz})^2\right)^{-1}$$
$$= D_{\text{tree}} + ig \int_z \left[D_{z1}D_{z2}D_{z3}D_{z4} + \tfrac{1}{2} D_{ab}D_{zc}D_{zd}D_{zz}\right] + \mathcal{O}(g^2)$$

- the same terms as in canonical quantisation