1. Quantum Field Theory (QFT) — Approaches

Classical Field Theory

- for the classic (non-quantum, real, free) Klein-Gordon field

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \tfrac{1}{2}m^2\phi^2 = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) - \tfrac{1}{2}m^2\phi^2$$

- field equations are the Euler-Lagrange equations:

$$0 = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} \stackrel{\text{here}}{=} \partial_\mu(\partial^\mu\phi) + m^2\phi = (\partial^2 + m^2)\phi$$

- they are fulfilled by plain wave solutions $f_{\vec{k}}(t,\vec{x}) = \frac{e^{-ik.x}}{(2\pi)^{3/2}(2E_{\vec{k}})^{1/2}}$
  - $E_{\vec{k}}^2 = \vec{k}^2 + m^2$
  - $k.x = k_\mu x^\mu = E_{\vec{k}}t - \vec{k}\cdot\vec{x}$

- a general solution is a superposition (Fourier integral)

$$\phi(t,\vec{x}) = \int d^3\vec{k}\left[a(\vec{k})f_{\vec{k}}(t,\vec{x}) + a^\dagger(\vec{k})f^*_{\vec{k}}(t,\vec{x})\right]$$

Canonical Quantisation

- in Quantum Mechanics we started with
  - generalized coordinates q_i
  - conjugate momenta $p_i = \frac{\partial L}{\partial \dot{q}}$
  - equal time commutation relations

$$[q_j, q_k] = [p_j, p_k] = 0 \qquad \text{and} \qquad [q_j, p_k] = i\hbar\delta_{jk}$$

- applying these to the harmonic oscillator
  - states $|n\rangle$ have n quanta
  - introducing creation operators $a^\dagger$ and annihilation operators a:

$$a^\dagger|n\rangle = c_+|n+1\rangle \qquad \text{and} \qquad a|n\rangle = c_-|n-1\rangle$$

  - have also equal time commutation relations

$$[a^\dagger, a^\dagger] = [a, a] = 0 \qquad \text{and} \qquad [a, a^\dagger] = 1$$

Canonical Quantisation

- the quantum system is given by a quantum state $|\Psi\rangle$
- it can be labeled by the eigenvalues of operators:
  - the position of Ψ is found by $\vec{X}|\Psi\rangle = \vec{x}_\Psi|\Psi\rangle$
  - the momentum of Ψ is found by $\vec{P}|\Psi\rangle = \vec{p}_\Psi|\Psi\rangle$
  - the space wave-function of Ψ is $\Psi(\vec{x}) = \langle\vec{x}|\Psi\rangle$
  - the momentum wave-function of Ψ is $\widetilde{\Psi}(\vec{p}) = \langle\vec{p}|\Psi\rangle$
  - and $\langle\vec{x}|\vec{p}\rangle = e^{i\vec{p}.\vec{x}}$ facilitates the Fourier transform

$$\Psi(\vec{x}) = \langle\vec{x}|\Psi\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\langle\vec{x}|\vec{p}\rangle\langle\vec{p}|\Psi\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}.\vec{x}}\,\widetilde{\Psi}(\vec{p})$$

and

$$\widetilde{\Psi}(\vec{p}) = \langle\vec{p}|\Psi\rangle = \int d^3\vec{x}\,\langle\vec{p}|\vec{x}\rangle\langle\vec{x}|\Psi\rangle = \int d^3\vec{x}\; e^{-i\vec{p}.\vec{x}}\,\Psi(\vec{x})$$

- time evolution is given by the Hamiltonian:

$$|\Psi(t_1)\rangle = e^{-iH(t_1-t_0)}|\Psi(t_0)\rangle$$

Canonical Quantisation

defining asymptotic **in-** and **out**-states $|\Psi_\alpha^\pm\rangle$ as

- α denotes all defining quantum numbers:
  - momentum, spin, particle-type, etc.

- eigenstates of the full Hamiltonian: $H|\Psi_\alpha^\pm\rangle = E_\alpha|\Psi_\alpha^\pm\rangle$
- non-interacting $\Rightarrow$ like eigenstates of the free Hamiltonian H_0:

$$H_0|\Phi_\alpha\rangle = E_\alpha|\Phi_\alpha\rangle$$

- though the eigenvalues are the same, the states are not: $|\Psi_\alpha^\pm\rangle \neq |\Phi_\alpha\rangle$
- the states are orthogonal:

$$\langle\Psi_\beta^\pm|\Psi_\alpha^\pm\rangle = \langle\Phi_\beta|\Phi_\alpha\rangle = \delta(\beta-\alpha)$$

- the states are complete:

$$\mathbf{1} = \int d\alpha\,|\Psi_\alpha^\pm\rangle\langle\Psi_\alpha^\pm| = \int d\alpha\,|\Phi_\alpha\rangle\langle\Phi_\alpha|$$

Canonical Quantisation

- splitting the Hamiltonian into a free part H_0 and an interaction part V

$$H = H_0 + V \qquad \Rightarrow \qquad (E_\alpha - H_0)|\Psi_\alpha^\pm\rangle = V|\Psi_\alpha^\pm\rangle$$

- without the interaction V, $|\Psi_\alpha^\pm\rangle \sim |\Phi_\alpha\rangle$
- using the completeness of $|\Phi_\alpha\rangle$
- we get the Lippmann-Schwinger equations

$$|\Psi_\alpha^\pm\rangle = |\Phi_\alpha\rangle + (E_\alpha - H_0 \pm i\epsilon)^{-1} V|\Psi_\alpha^\pm\rangle = |\Phi_\alpha\rangle + \int d\beta \frac{|\Phi_\beta\rangle T_{\beta\alpha}^\pm}{E_\alpha - E_\beta \pm i\epsilon}$$

  where $T_{\beta\alpha}^\pm = \langle\Phi_\beta|V|\Psi_\alpha^\pm\rangle$.

- the S-matrix is defined as $S_{\beta\alpha} = \langle\Psi_\beta^+|\Psi_\alpha^-\rangle$
- the S-matrix is unitary due to the completeness of states:

$$\int d\beta\, S_{\beta\gamma}^* S_{\beta\alpha} = \int d\beta\, \langle\Psi_\gamma^-|\Psi_\beta^+\rangle\langle\Psi_\beta^+|\Psi_\alpha^-\rangle = \langle\Psi_\gamma^-|\Psi_\alpha^-\rangle = \delta(\gamma - \alpha)$$

Canonical Quantisation

For Quantum Field Theory we start with

- generalized coordinates $q_i \to \phi(t,\vec{x})$ (or other fields)
- conjugate momenta $p_i \to \pi(t,\vec{x}) = \frac{\partial \mathcal{L}(t,\vec{x})}{\partial \dot{\phi}(t,\vec{x})}$
- equal time commutation relations

$$[\phi(t,\vec{x}),\phi(t,\vec{y})] = [\pi(t,\vec{x}),\pi(t,\vec{y})] = 0$$

$$\text{and} \qquad [\phi(t,\vec{x}),\pi(t,\vec{y})] = i\hbar\delta^3(\vec{x}-\vec{y})$$

- these give for the Fourier components $a(\vec{k})$ and $a^\dagger(\vec{k})$

$$[a^\dagger(t,\vec{k}),a^\dagger(t,\vec{k}')] = [a(t,\vec{k}),a(t,\vec{k}')] = 0$$

$$\text{and} \qquad [a(t,\vec{k}),a^\dagger(t,\vec{k}')] = \delta^3(\vec{k}-\vec{k}')$$

- the same as the harmonic oscillator
  $\Rightarrow$ $a^\dagger(\vec{k})$ and $a(\vec{k})$ are creation- and annihilation operators.

Canonical Quantisation

defining the Hilbert space as a complete set of states:

- the vacuum $|0\rangle$ with $\langle 0|0\rangle = 1$
- $|n_i(\vec{k})\rangle$ with n particles of type i and momentum $\vec{k}$
  - created by $a_i^\dagger(\vec{k})$: $a_i^\dagger(\vec{k})|n_i(\vec{k})\rangle = c_{n+}(\vec{k})|(n+1)_i(\vec{k})\rangle$
  - writing the one-particle states as $|\vec{k}_i\rangle = \sqrt{2E_{\vec{k}}}\,a_i^\dagger(\vec{k})|0\rangle$
  - we have the normalisation (setting $\hbar = 1$)

$$\begin{aligned}\langle \vec{k}'_i|\vec{k}_j\rangle &= 2\sqrt{E_{\vec{k}'}E_{\vec{k}}}\,\langle 0|a_i(\vec{k}')a_j^\dagger(\vec{k})|0\rangle = 2\sqrt{E_{\vec{k}'}E_{\vec{k}}}\,\langle 0|[a_i(\vec{k}'),a_j^\dagger(\vec{k})]|0\rangle \\ &= 2\sqrt{E_{\vec{k}'}E_{\vec{k}}}\,\langle 0|\delta_{ij}\delta^3(\vec{k}-\vec{k}')|0\rangle = 2\sqrt{E_{\vec{k}}E_{\vec{k}}}\,\delta_{ij}\delta^3(\vec{k}-\vec{k}')\langle 0|0\rangle \\ &= 2E_{\vec{k}}\,\delta_{ij}\delta^3(\vec{k}-\vec{k}')\end{aligned}$$

- since it is a complete set, we can write

$$1 = \sum_{n,i}\int \frac{d^3\vec{k}}{(2\pi)^3}\;|n_i(\vec{k})\rangle\frac{1}{2E_{\vec{k}}}\langle n_i(\vec{k})|$$

Canonical Quantisation

- from the Lagrangian we get the Hamiltonian $H = \int d^3\vec{x}\,\mathcal{H}$ with

$$\mathcal{H}(t,\vec{x}) = \pi(t,\vec{x})\dot{\phi}(t,\vec{x}) - \mathcal{L}(t,\vec{x}) \overset{\text{here}}{=} \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) + \tfrac{1}{2}m^2\phi^2$$

- using

$$\phi(t,\vec{x}) = \int d^3\vec{k}\left[a(\vec{k})f_{\vec{k}}(t,\vec{x}) + a^\dagger(\vec{k})f^*_{\vec{k}}(t,\vec{x})\right]$$

- and the equal time commutator, we get

$$H = \int d^3\vec{k}\,\tfrac{1}{2}E_{\vec{k}}\left[a(\vec{k})a^\dagger(\vec{k}) + a^\dagger(\vec{k})a(\vec{k})\right] = \int d^3\vec{k}\,E_{\vec{k}}[N_{\vec{k}} + \tfrac{1}{2}\delta^3(\vec{0})]$$

- that yields a zero-point energy of $E_0 = \frac{1}{2}\delta^3(\vec{0})\int d^3\vec{k}\,E_{\vec{k}} \sim \infty$
- normal ordering $N\{H\}$ gets rid of the zero-point energy:
  - $a^\dagger$ is always left of a: $N\{a(\vec{k})a^\dagger(\vec{k})\} := a^\dagger(\vec{k})a(\vec{k})$

Canonical Quantisation ... considering propagation:

- the correlator is the amplitude of a field between x and y

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int d^3\vec{k}\, \frac{e^{-ik.(x-y)}}{(2\pi)^3 2E_{\vec{k}}}$$

- the commutator becomes

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = D(x-y) - D(y-x) = \int d^3\vec{k}\, \frac{e^{-ik.(x-y)} - e^{ik.(x-y)}}{(2\pi)^3 2E_{\vec{k}}}$$

- the Feynman propagator is

$$D_F(x-y) = \begin{cases} D(x-y) & \text{for} \quad x^0 > y^0 \\ D(y-x) & \text{for} \quad x^0 < y^0 \end{cases} \qquad \equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$$

  with T denoting time ordering

- or in the usual notation

$$D_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik.(x-y)}$$

Path Integral Quantisation

- the amplitude for $|\Psi_I\rangle$ going to $|\Psi_F\rangle$ was

$$\mathcal{A} = \langle \Psi_F | e^{-iH(t_F - t_I)} | \Psi_I \rangle$$

- splitting the time interval into N equal parts δt
- inserting complete set of states $|q_n\rangle$ at time $t_I + n\delta t$

$$\mathcal{A} = \left[\prod_{n=0}^{N} \int dq_n \right] \langle \Psi_F | q_N \rangle \langle q_N | e^{-iH\delta t} | q_{N-1} \rangle \ldots \langle q_1 | e^{-iH\delta t} | q_0 \rangle \langle q_0 | \Psi_I \rangle$$

- inserting complete set of states $|p_n\rangle$ we get

$$\begin{aligned} \langle q_{n+1} | e^{-iH\delta t} | q_n \rangle &= \int dp \, \langle q_{n+1} | e^{-iH\delta t} | p \rangle \langle p | q_n \rangle = \int dp \, e^{-ih\delta t} \langle q_{n+1} | p \rangle \langle p | q_n \rangle \\ &= \int dp \, e^{-i[\frac{p^2}{2m} + V(q)]\delta t} \, e^{ip(q_{n+1} - q_n)} \end{aligned}$$

where h is no longer an operator, but only a normal function.

Path Integral Quantisation

- integrating the gaussian integral $\int_{-\infty}^{\infty} dx e^{\frac{i}{2}ax^2+iJx} = \sqrt{\frac{2\pi i}{a}} e^{-\frac{i}{2a}J^2}$

$$\begin{aligned}\langle q_{n+1}|e^{-iH\delta t}|q_n\rangle &= e^{-iV(q)\delta t}\sqrt{\frac{2\pi i m}{\delta t}}\, e^{i\frac{m}{2\delta t}(q_{n+1}-q_n)^2} \\ &\overset{\delta t\to 0}{=} \sqrt{\frac{2\pi i m}{\delta t}}\, e^{i(\frac{m}{2}\dot{q}_n^2-V(q))\delta t} = \sqrt{\frac{2\pi i m}{\delta t}}\, e^{iL(q_n,\dot{q}_n)\delta t}\end{aligned}$$

- inserting this into the amplitude

$$\mathcal{A} = \int dq_N\, \langle\Psi_F|q_N\rangle \left[\prod_{n=0}^{N-1} \sqrt{\frac{2\pi i m}{\delta t}} \int dq_n\right] \prod_{n=0}^{N-1} e^{iL(q_n,\dot{q}_n)\delta t} \langle q_0|\Psi_I\rangle$$

- in the limit $\delta t \to 0$ we get the initial and the final state wavefunctions:

$$\langle\Psi_F|q_N\rangle \to \Psi_F(q_N)^* \qquad \langle q_0|\Psi_I\rangle \to \Psi_I(q_0)$$

- taking initial and final state as the groundstate $|0\rangle$

$$Z = \langle 0|e^{-iH(t_F-t_I)}|0\rangle = \int \mathcal{D}q(t)\, e^{\frac{i}{\hbar}\int_{t_I}^{t_F} dt L(q(t),\dot{q}(t))}$$

Path Integral Quantisation

- making the step to fields:

q	$\to$	ϕ	$q_i(t)$	$\to$	$\phi(t,\vec{x}) = \phi(x)$
i	$\to$	$\vec{x}$	$\sum_i$	$\to$	$\int d^D\vec{x}$

- sending $t_I \to -\infty$ and $t_F \to \infty$
- including sources $J(x)$ for the field $\phi(x)$
- we get the Path Integral

$$Z(J) = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int d^Dx \mathcal{L}(\phi,\partial\phi)+J\phi} \overset{\hbar\to 1}{=} \int \mathcal{D}\phi \, e^{i(S+\int_x J\phi)}$$

- this includes all quantum corrections !