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Using the Pathintegral to derive the Feynman Rules for the ABC-Theory

notes for the lecture Modernioji Teorinė Fizika

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1 Introduction

This note is written mainly for students of the course *Modernoji Teorinė Fizika* at VU. The goal of these notes is to illustrate how the methods of Quantum Field Theory (QFT) are used to obtain the Feynman rules for the *ABC-Theory*, which is in a certain way the simplest possible toy model in 3+1 dimensions. The *ABC-Theory* describes three interacting scalar fields, that have only a singe intereaction. It was introduced for teaching purposes by David Griffiths [1] and then extensively used by I. J. R. Aitchison and A. J. G. Hey [2] to explain preturbative expansions, regularization and renormalisation in QFT.

2 Lagrangian and action

The simplest quantity in space is a point without any structure, i.e. the real scalar. So let us first consider the Lagrangian of a single real scalar field $\phi(x)$ without interactions:

$$\mathcal{L}_\phi = \tfrac{1}{2}\left(\partial^\mu \phi(x)\right)\left(\partial_\mu \phi(x)\right) - \tfrac{1}{2} m_\phi^2 \phi^2(x) \ . \tag{1}$$

The action is the integral over the Lagrangian. So the action S_ϕ of the free, i.e. non interacting, real scalar field $\phi = \phi(x)$, can be written as

$$\begin{aligned} S_\phi &= \int d^4x \, \mathcal{L}_\phi = \int d^4x \, \tfrac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \tfrac{1}{2} m_\phi^2 \phi^2 \\ &= -\tfrac{1}{2} \int d^4x \, \phi(\partial^2 + m_\phi^2)\phi + \tfrac{1}{2}[\phi(\partial_\mu \phi)]_{\text{at } x\to\infty} \ , \end{aligned} \tag{2}$$

where we used partial integration. The last piece can usually be set to zero, as fields that do not vanish rapidly enough at infinity are not localized enough to be treated as particles.

Using the Fourier transformation on the field ϕ

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\phi}(k) \quad \text{and} \quad \tilde{\phi}(k) = \int d^4x e^{ikx} \phi(x) \tag{3}$$

with the understanding, that $kx = k_\mu x^\mu$, we can rewrite the action

$$\begin{aligned} S_\phi &= \int d^4x \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \tfrac{1}{2}(\partial^\mu e^{-ikx}\tilde{\phi}(k))(\partial_\mu e^{-ipx}\tilde{\phi}(p)) - \tfrac{1}{2} m_\phi^2 e^{-ikx}\tilde{\phi}(k) e^{-ipx}\tilde{\phi}(p) \\ &= \tfrac{1}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} [(-ik^\mu)(-ip_\mu) - m_\phi^2] e^{-i(k+p)x} \tilde{\phi}(k)\tilde{\phi}(p) \ . \end{aligned} \tag{4}$$

The integral over x gives the Dirac delta function $\delta(k+p)$, which can be used to perform the k-integral by replacing k with $(-p)$:

$$\begin{aligned} S_\phi &= -\tfrac{1}{2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \tilde{\phi}(k)[k^\mu p_\mu + m_\phi^2](2\pi)^4 \delta(k+p)\tilde{\phi}(p) \\ &= -\tfrac{1}{2} \int \frac{d^4p}{(2\pi)^4} \tilde{\phi}(-p)(-p^2 + m_\phi^2)\tilde{\phi}(p) \ . \end{aligned} \tag{5}$$

When Fourier transforming back to $\phi(x)$

$$\begin{aligned} S_\phi &= -\tfrac{1}{2}\int d^4x \int d^4y \int \frac{d^4p}{(2\pi)^4} e^{i(-p)x}\phi(x)(-p^2+m_\phi^2)e^{ipy}\phi(y) \\ &= -\tfrac{1}{2}\int d^4x \int d^4y\, \phi(x)\left[\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}(-p^2+m_\phi^2)\right]\phi(y) \ . \end{aligned} \tag{6}$$

we could go back to the same for where we started, but we can also use the opportunity to define the Feynman propagator

$$D_F(x,y) := D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_\phi^2+i\epsilon}e^{-ip(x-y)} \ , \tag{7}$$

where the infinitesimal $i\epsilon$ gives the prescription, how we should calculate the integral, when $p^2 \to m_\phi^2$. Its inverse $D_F^{-1}(x,y)$ can be found from

$$\delta(x-z) = \int d^4y\, D_F(x,y)D_F^{-1}(y,z) = \int d^4y\, D_F^{-1}(x,y)D_F(y,z) \tag{8}$$

with the ansatz

$$D_F^{-1}(x,y) = \int \frac{d^4q}{(2\pi)^4}e^{-iq(x-y)}\tilde{d}(q) \tag{9}$$

and the help of Fourier transformations

$$\begin{aligned} e^{ikz} &= \int d^4x\, e^{ikx}\delta(x-z) \\ &= \int d^4x\, e^{ikx}\int d^4y \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_\phi^2+i\epsilon}e^{-ip(x-y)}\int \frac{d^4q}{(2\pi)^4}e^{-iq(y-z)}\tilde{d}(q) \\ &= \int \frac{d^4p}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4}\int d^4x\, e^{i(k-p)x}\int d^4y\, e^{i(p-q)y}\frac{i}{p^2-m_\phi^2+i\epsilon}e^{iqz}\tilde{d}(q) \\ &= \int \frac{d^4p}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4}(2\pi)^4\delta(k-p)(2\pi)^4\delta(p-q)\frac{i}{p^2-m_\phi^2+i\epsilon}e^{iqz}\tilde{d}(q) \\ &= \frac{ie^{ikz}\tilde{d}(k)}{k^2-m_\phi^2+i\epsilon} \ , \end{aligned} \tag{10}$$

or simply $\tilde{d}(k) = -i(k^2-m_\phi^2+i\epsilon)$, where the $i\epsilon$ is no longer necessary, since the function does not have any singularity. So we can write

$$D_F^{-1}(z) = \int \frac{d^4q}{(2\pi)^4}e^{-iqz}\tilde{d}(q) = i\int \frac{d^4q}{(2\pi)^4}e^{-iqz}(-k^2+m_\phi^2) \ . \tag{11}$$

Here is also the best place to note, that $D_F(x,y)$ and $D_F^{-1}(x,y)$ are symmetric under the exchange of x and y:

$$\begin{aligned} D_F(y,x) &= \int_{-\infty}^{+\infty}\frac{d^Dp}{(2\pi)^4}\frac{ie^{-ip(y-x)}}{p^2-m_\phi^2+i\epsilon} = (-1)^D\int_{+\infty}^{-\infty}\frac{d^Dk}{(2\pi)^4}\frac{ie^{ik(y-x)}}{(-k)^2-m_\phi^2+i\epsilon} \\ &= (-1)^{2D}\int_{-\infty}^{+\infty}\frac{d^Dk}{(2\pi)^4}\frac{ie^{-ik(x-y)}}{k^2-m_\phi^2+i\epsilon} = (-1)^{2D}D_F(x,y) \ , \end{aligned} \tag{12}$$

for all integer dimensions D.

For the action we can now write

$$\begin{aligned} S_\phi &= \tfrac{1}{2}\int d^4x \int d^4y\, \phi(x) i \left[i \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}(-p^2+m_\phi^2)\right]\phi(y) \\ &= \tfrac{1}{2}\int d^4x \int d^4y\, \phi(x) i D_F^{-1}(x-y)\phi(y) = \tfrac{1}{2}\phi_x (iD_F^{-1})_{xy}\phi_y \ , \end{aligned} \tag{13}$$

where the last rewriting of the action emphasizes that it is bilinear in the scalar field.

3 Pathintegral

The pathintegral describes the vacuum to vacuum transition by summing over all quantum mechanically possible trajectories in the presence of a source function $J(x)$. This generating functional can now be written as

$$Z(J) = \int \mathcal{D}\phi \, \exp\{iS_\phi + i\int d^4x \, J(x)\phi(x)\} = \int \mathcal{D}\phi \, e^{-\frac{1}{2}\phi_x (D_F^{-1})_{xy}\phi_y + iJ_x\phi_x} \ , \tag{14}$$

which is a Gaussian integral of the form

$$\int_{-\infty}^{+\infty} e^{-bx^2+cx} dx = \sqrt{\frac{\pi}{b}} e^{\frac{c^2}{4b}} \tag{15}$$

with $b = \frac{1}{2}$ and $c = iJ$. So

$$Z(J) = N \times e^{-\frac{1}{2}J_x (D_F)_{xy} J_y} = N \times e^{\frac{i}{2}J_a (iD_F)_{ab} J_b} \ , \tag{16}$$

with an undetermined normalization factor N. When moving to the logarithm

$$W(J) := -i \ln Z(J) = -i(\ln N + \tfrac{i}{2} J_a (iD_F)_{ab} J_b) \tag{17}$$

we find that the propagator is exactly the second functional derivative of W:

$$\begin{aligned} &\frac{\delta}{\delta J_x}\frac{\delta}{\delta J_y} W(J) \\ &= \frac{\delta}{\delta J_x}\frac{\delta}{\delta J_y}(-i\ln N + \tfrac{1}{2}J_a(iD_F)_{ab}J_b) = \frac{\delta}{\delta J_x}(0 + \tfrac{1}{2}\delta_a^y (iD_F)_{ab}J_b + \tfrac{1}{2}J_a(iD_F)_{ab}\delta_b^y) \\ &= \tfrac{1}{2}((iD_F)_{yb}\delta_b^x + \delta^x(iD_F)_{ay}) = \tfrac{1}{2}((iD_F)_{yx} + (iD_F)_{xy}) = (iD_F)_{xy} \end{aligned} \tag{18}$$

So we get an **exact** description of a **non-interacting** theory.

Including now an interaction

$$\mathcal{L}_I = gABC \tag{19}$$

of three different scalar fields A, B, and C, which are in their free form identical to our previous field ϕ, we get the total action

$$\begin{aligned} S &= S_A + S_B + S_C + S_I \\ &= \tfrac{1}{2}A_x(iD_{FA}^{-1})_{xy}A_y + \tfrac{1}{2}B_x(iD_{FB}^{-1})_{xy}B_y + \tfrac{1}{2}C_x(iD_{FC}^{-1})_{xy}C_y + g\int_z A_z B_z C_z \end{aligned} \tag{20}$$

and the pathintegral

$$Z(J_A, J_B, J_C; g) = \int \mathcal{D}A\,\mathcal{D}B\,\mathcal{D}C\ \exp\{iS + iJ_{Ax}A_x + iJ_{Bx}B_x + iJ_{Cx}C_x\}\ . \tag{21}$$

For zero coupling, i.e. $g = 0$, we can still solve this integral exactly

$$\begin{aligned} &Z(J_A, J_B, J_C; 0) \\ &= \int \mathcal{D}A\,\mathcal{D}B\,\mathcal{D}C\ \exp\{iS_A + iJ_{Ax}A_x + iS_B + iJ_{Bx}B_x + iS_C + iJ_{Cx}C_x\} \\ &= N \times \exp\{\tfrac{i}{2}J_{Ax}(iD_{FA})_{xy}J_{Ay} + \tfrac{i}{2}J_{Bx}(iD_{FB})_{xy}J_{By} + \tfrac{i}{2}J_{Cx}(iD_{FC})_{xy}J_{Cy}\}\ . \end{aligned} \tag{22}$$

Taking the interaction term and writing the exponential as its powerseries

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{23}$$

and noting that we can write

$$\int \mathcal{D}A\, A(z) \exp\{iS_A + iJ_{Ax}A_x\} = \int \mathcal{D}A\, \frac{\delta}{i\delta J_{Az}} \exp\{iS_A + iJ_{Ax}A_x\} \tag{24}$$

we can reformulate the pathintegral as $Z(J_A, J_B, J_C; g)$

$$\begin{aligned} &= \int \mathcal{D}A\,\mathcal{D}B\,\mathcal{D}C\ \exp\{iS_I\} \times \exp\{iS_A + iJ_{Ax}A_x + iS_B + iJ_{Bx}B_x + iS_C + iJ_{Cx}C_x\} \\ &= \int \mathcal{D}A\,\mathcal{D}B\,\mathcal{D}C\ \left[\sum_{k=0}^{\infty} \frac{1}{k!}\left(ig\int_z A_z B_z C_z\right)^k\right] \times e^{iS_A + iJ_{Ax}A_x + iS_B + iJ_{Bx}B_x + iS_C + iJ_{Cx}C_x} \\ &= \int \mathcal{D}A\,\mathcal{D}B\,\mathcal{D}C\ \left[\sum_{k=0}^{\infty} \frac{1}{k!}\left(ig\int_z \frac{\delta}{i\delta J_{Az}}\frac{\delta}{i\delta J_{Bz}}\frac{\delta}{i\delta J_{Cz}}\right)^k\right] \times e^{iS_A + iJ_{Ax}A_x + iS_B + iJ_{Bx}B_x + iS_C + iJ_{Cx}C_x} \\ &= \left[\sum_{k=0}^{\infty} \frac{1}{k!}\left(-g\int_z \frac{\delta}{\delta J_{Az}}\frac{\delta}{\delta J_{Bz}}\frac{\delta}{\delta J_{Cz}}\right)^k\right] \times \int \mathcal{D}A\,\mathcal{D}B\,\mathcal{D}C\, e^{iS_A + iJ_{Ax}A_x + iS_B + iJ_{Bx}B_x + iS_C + iJ_{Cx}C_x} \\ &= \exp\left\{-g\int_z \frac{\delta^3}{\delta J_{Az}\,\delta J_{Bz}\,\delta J_{Cz}}\right\} \times Z(J_A, J_B, J_C; 0)\ . \end{aligned} \tag{25}$$

4 Generating Functionals

When we go to the logarithm we can no longer split the exponents and solve exactly, as $\frac{\delta}{\delta J_{Az}}$ does not commute with $J_{Ax}(iD_{FA})_{xy}J_{Ay}$ and has only a meaning when it acts on the functional.

But it still allows the **definition** of the full propagator of the interacting theory as

$$(iD_F)_{xy} := \frac{\delta}{\delta J_x}\frac{\delta}{\delta J_y} W(J; g)\ . \tag{26}$$

Therefore $W(J;g)$ is also called the generating functional of all connected Greensfunctions. We will write out the full propagator with the definition of the pathintegral later and see, that it contains the averaged correlator of two fields at the points x and y: $\langle\phi(x)\phi(y)\rangle$.

As the next step we define the classical field

$$\begin{aligned} \varphi(x) &:= \frac{\delta}{\delta J(x)} W(J;g) = \frac{\delta}{\delta J_x}[-i \ln Z(J;g)] = \frac{-i}{Z}\frac{\delta Z(J;g)}{\delta J_x} \\ &= -i \frac{\int \mathcal{D}\phi \;\; i\phi_x e^{iS_\phi + iJ_x\phi_x}}{\int \mathcal{D}\phi \;\; e^{iS_\phi + iJ_x\phi_x}} = \langle \phi_x \rangle \ , \end{aligned} \tag{27}$$

which is the average over all quantum states generated by the classical source function $J(x)$. It allows to perform the Legendre transform $W(J) \to \Gamma(\varphi)$ from the generating functional of all connected Greensfunctions to the generating functional of all one-particle-irreducible (1PI) vertices

$$\Gamma := W - J_x\varphi_x = \Gamma(\varphi;g) \ . \tag{28}$$

which is also called **effective potential**.

We will first investigate some features of and relations between the two generating functionals. The first is the normal property of the Legendre transformation:

$$\begin{aligned} \frac{\delta}{\delta\varphi(x)}\Gamma(\varphi;g) &= \frac{\delta}{\delta\varphi_x}(W - J_z\varphi_z) = \frac{\delta}{\delta\varphi_x}W - (\frac{\delta}{\delta\varphi_x}J_z)\varphi_z - J_z(\frac{\delta}{\delta\varphi_x}\varphi_z) \\ &= \frac{\delta J_y}{\delta\varphi_x}\frac{\delta}{\delta J_y}W - (\frac{\delta J_z}{\delta\varphi_x})\varphi_z - J_z\delta_z^x = \frac{\delta J_y}{\delta\varphi_x}\varphi_y - \frac{\delta J_z}{\delta\varphi_x}\varphi_z - J_x = -J_x \end{aligned} \tag{29}$$

But this property has important implications. One is, that the effective potential is as a functional an extremum in the classical field when there is no external source. This is also reflected by the Euler Lagrange equations: the classical solution is the minmum of the action. And it holds also with the inclusion of quantum corrections for the effective action — hence the name.

There is also an important connection of the second derivatives of Γ and W. Taking the above equation and performing the derivative with respect to the source we get

$$\begin{aligned} \delta_x^z &= \frac{\delta}{\delta J_z}J_x = \frac{\delta}{\delta J_z}(-\frac{\delta}{\delta\varphi_x}\Gamma) = \frac{\delta\varphi_y}{\delta J_z}\frac{\delta}{\delta\varphi_y}(-\frac{\delta}{\delta\varphi_x}\Gamma) = \left(\frac{\delta}{\delta J_z}\frac{\delta W}{\delta J_y}\right)\left(-\frac{\delta^2\Gamma}{\delta\varphi_y\delta\varphi_x}\right) \\ &= \left(\frac{\delta^2 W}{\delta J_z \delta J_y}\right)\left(-\frac{\delta^2\Gamma}{\delta\varphi_y\delta\varphi_x}\right) \ , \end{aligned} \tag{30}$$

which basically states, that the second derivatives are inverse to each other:

$$\frac{\delta^2 W}{\delta J_x \delta J_y} = \left(-\frac{\delta^2\Gamma}{\delta\varphi_y\delta\varphi_x}\right)^{-1} \ , \tag{31}$$

and including the definition of the propagator

$$(iD_F)_{xy} = \frac{\delta^2 W}{\delta J_x \delta J_y} = \left(-\frac{\delta^2\Gamma}{\delta\varphi_y\delta\varphi_x}\right)^{-1} \ . \tag{32}$$

This allows us to write the derivative with respect to a source fully in terms of the effective action and the classical fields:

$$\frac{\delta}{\delta J_x} = \frac{\delta \varphi_y}{\delta J_x}\frac{\delta}{\delta \varphi_y} = (\frac{\delta}{\delta J_x}\frac{\delta W}{\delta J_y})\frac{\delta}{\delta \varphi_y} = (\frac{\delta^2 W}{\delta J_x \delta J_y})\frac{\delta}{\delta \varphi_y} = (iD_F)_{xy}\frac{\delta}{\delta \varphi_y} \quad , \tag{33}$$

or in the opposite direction

$$\begin{aligned}\frac{\delta}{\delta \varphi_x} &= \frac{\delta J_y}{\delta \varphi_x}\frac{\delta}{\delta J_y} = [\frac{\delta}{\delta \varphi_x}(-\frac{\delta \Gamma}{\delta \varphi_y})]\frac{\delta}{\delta J_y} = (-\frac{\delta^2 \Gamma}{\delta \varphi_x \delta \varphi_y})\frac{\delta}{\delta J_y} = [-(iD_F)^{-1}_{xy}]\frac{\delta}{\delta J_y} \\ &= (iD_F^{-1})_{xy}\frac{\delta}{\delta J_y} \quad . \end{aligned} \tag{34}$$

As the next step we investigate the connection of the higher derivatives of W, Z and Γ. From the direct writing of the pathintegral we get:

$$\begin{aligned}\frac{i\delta^2}{\delta J_x \delta J_y}W &= \frac{i\delta^2}{\delta J_x \delta J_y}[-i\ln Z] = \frac{\delta}{\delta J_x}[\frac{1}{Z}\frac{\delta Z}{\delta J_y}] = \frac{1}{Z}\frac{\delta^2 Z}{\delta J_x \delta J_y} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_x}\frac{\delta Z}{\delta J_y} \\ &= [\langle \phi_x \phi_y \rangle - \langle \phi_x \rangle \langle \phi_y \rangle] = \langle \phi_x \phi_y \rangle_{connected} \end{aligned} \tag{35}$$

and

$$\begin{aligned}\frac{i\delta^3}{\delta J_x \delta J_y \delta J_z}W &= \frac{\delta}{\delta J_x}[\frac{1}{Z}\frac{\delta^2 Z}{\delta J_y \delta J_z} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_y}\frac{\delta Z}{\delta J_z}] \\ &= \frac{1}{Z}\frac{\delta^3 Z}{\delta J_x \delta J_y \delta J_z} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_x}\frac{\delta^2 Z}{\delta J_y \delta J_z} - \frac{1}{Z^2}\frac{\delta^2 Z}{\delta J_x \delta J_y}\frac{\delta Z}{\delta J_z} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_y}\frac{\delta^2 Z}{\delta J_x \delta J_z} \\ &\quad +2\frac{1}{Z^3}\frac{\delta Z}{\delta J_x}\frac{\delta Z}{\delta J_y}\frac{\delta Z}{\delta J_z} \\ &= [\langle \phi_x \phi_y \phi_z \rangle - \langle \phi_x \rangle \langle \phi_y \phi_z \rangle - \langle \phi_x \phi_y \rangle \langle \phi_z \rangle - \langle \phi_y \rangle \langle \phi_x \phi_z \rangle + 2\langle \phi_x \rangle \langle \phi_y \rangle \langle \phi_z \rangle] \\ &= \langle \phi_x \phi_y \phi_z \rangle_{connected} \end{aligned} \tag{36}$$

and

$$\begin{aligned}&\frac{i\delta^4}{\delta J_w \delta J_x \delta J_y \delta J_z}W \\ &= \frac{\delta}{\delta J_w}[\frac{1}{Z}\frac{\delta^3 Z}{\delta J_x \delta J_y \delta J_z} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_x}\frac{\delta^2 Z}{\delta J_y \delta J_z} - \frac{1}{Z^2}\frac{\delta^2 Z}{\delta J_x \delta J_y}\frac{\delta Z}{\delta J_z} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_y}\frac{\delta^2 Z}{\delta J_x \delta J_z} + 2\frac{1}{Z^3}\frac{\delta Z}{\delta J_x}\frac{\delta Z}{\delta J_y}\frac{\delta Z}{\delta J_z}] \\ &= \frac{1}{Z}\frac{\delta^4 Z}{\delta J_w \delta J_x \delta J_y \delta J_z} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_x}\frac{\delta^3 Z}{\delta J_w \delta J_y \delta J_z} - \frac{1}{Z^2}\frac{\delta^3 Z}{\delta J_w \delta J_x \delta J_y}\frac{\delta Z}{\delta J_z} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_y}\frac{\delta^3 Z}{\delta J_w \delta J_x \delta J_z} \\ &\quad -\frac{1}{Z^2}\frac{\delta^2 Z}{\delta J_w \delta J_x}\frac{\delta^2 Z}{\delta J_y \delta J_z} - \frac{1}{Z^2}\frac{\delta^2 Z}{\delta J_x \delta J_y}\frac{\delta^2 Z}{\delta J_w \delta J_z} - \frac{1}{Z^2}\frac{\delta^2 Z}{\delta J_w \delta J_y}\frac{\delta^2 Z}{\delta J_x \delta J_z} \\ &\quad +2\frac{1}{Z^3}\frac{\delta^2 Z}{\delta J_w \delta J_x}\frac{\delta Z}{\delta J_y}\frac{\delta Z}{\delta J_z} + 2\frac{1}{Z^3}\frac{\delta Z}{\delta J_x}\frac{\delta^2 Z}{\delta J_w \delta J_y}\frac{\delta Z}{\delta J_z} + 2\frac{1}{Z^3}\frac{\delta Z}{\delta J_x}\frac{\delta Z}{\delta J_y}\frac{{}^2\delta Z}{\delta J_w \delta J_z} \\ &\quad -\frac{1}{Z^2}\frac{\delta Z}{\delta J_w}\frac{\delta^3 Z}{\delta J_x \delta J_y \delta J_z} + 2\frac{1}{Z^3}\frac{\delta Z}{\delta J_w}[\frac{\delta Z}{\delta J_x}\frac{\delta^2 Z}{\delta J_y \delta J_z} + \frac{\delta^2 Z}{\delta J_x \delta J_y}\frac{\delta Z}{\delta J_z} + \frac{\delta Z}{\delta J_y}\frac{\delta^2 Z}{\delta J_x \delta J_z}] \\ &\quad -6\frac{1}{Z^4}\frac{\delta Z}{\delta J_w}\frac{\delta Z}{\delta J_x}\frac{\delta Z}{\delta J_y}\frac{\delta Z}{\delta J_z} \end{aligned} \tag{37}$$

which gives

$$\begin{aligned}
&\frac{i\delta^4}{\delta J_w \delta J_x \delta J_y \delta J_z} W \\
&= \langle\phi_w\phi_x\phi_y\phi_z\rangle - \langle\phi_w\rangle\langle\phi_x\phi_y\phi_z\rangle - \langle\phi_x\rangle\langle\phi_w\phi_y\phi_z\rangle - \langle\phi_y\rangle\langle\phi_w\phi_x\phi_z\rangle - \langle\phi_z\rangle\langle\phi_w\phi_x\phi_y\rangle \\
&\qquad -\langle\phi_w\phi_x\rangle\langle\phi_y\phi_z\rangle - \langle\phi_w\phi_y\rangle\langle\phi_x\phi_z\rangle - \langle\phi_w\phi_z\rangle\langle\phi_x\phi_y\rangle \\
&\qquad +2\langle\phi_w\rangle\langle\phi_x\rangle\langle\phi_y\phi_z\rangle + 2\langle\phi_w\rangle\langle\phi_y\rangle\langle\phi_x\phi_z\rangle + 2\langle\phi_w\rangle\langle\phi_z\rangle\langle\phi_x\phi_y\rangle \\
&\qquad +2\langle\phi_x\rangle\langle\phi_y\rangle\langle\phi_w\phi_z\rangle + 2\langle\phi_x\rangle\langle\phi_z\rangle\langle\phi_w\phi_y\rangle + 2\langle\phi_y\rangle\langle\phi_z\rangle\langle\phi_w\phi_x\rangle \\
&\qquad -6\langle\phi_w\rangle\langle\phi_x\rangle\langle\phi_y\rangle\langle\phi_z\rangle \\
&= \langle\phi_w\phi_x\phi_y\phi_z\rangle_{connected} \qquad (38)
\end{aligned}$$

and so forth.

Interpreting the derivative with respect to the source as a starting point for a field, we can conclude that W generates diagrams where all starting points are connected and all the unconnected ones are substracted.

We can interpret the third derivative of W also in another way. We can write

$$\begin{aligned}
\frac{\delta^3}{\delta J_x \delta J_y \delta J_z} W &= \frac{\delta}{\delta J_x}\frac{\delta^2 W}{\delta J_y \delta J_z} = \frac{\delta}{\delta J_x}\left[\left(-\frac{-\delta^2\Gamma}{\delta\varphi_z\delta\varphi_y}\right)^{-1}\right] \\
&= (iD_F)_{xa}\frac{\delta}{\delta\varphi_a}\left[\left(\frac{\delta^2\Gamma}{\delta\varphi_z\delta\varphi_y}\right)^{-1}\right] \qquad (39)
\end{aligned}$$

which we can evaluate using the matrix identity $\partial M^{-1} = -M^{-1}(\partial M)M^{-1}$, obtained from

$$0 = \partial(1) = \partial(M\cdot M^{-1}) = (\partial M)M^{-1} + M(\partial M^{-1}) \ , \qquad (40)$$

and write

$$\begin{aligned}
\frac{\delta^3}{\delta J_x \delta J_y \delta J_z} W &= (iD_F)_{xa}\left(\frac{-\delta^2\Gamma}{\delta\varphi_z\delta\varphi_c}\right)^{-1}\left[-\frac{\delta}{\delta\varphi_a}\left(\frac{-\delta^2\Gamma}{\delta\varphi_c\delta\varphi_b}\right)\right]\left(\frac{-\delta^2\Gamma}{\delta\varphi_b\delta\varphi_y}\right)^{-1} \\
&= \left(\frac{-\delta^2\Gamma}{\delta\varphi_x\delta\varphi_a}\right)^{-1}\left(\frac{-\delta^2\Gamma}{\delta\varphi_y\delta\varphi_b}\right)^{-1}\left(\frac{-\delta^2\Gamma}{\delta\varphi_z\delta\varphi_c}\right)^{-1}\left(\frac{\delta^3\Gamma}{\delta\varphi_a\delta\varphi_b\delta\varphi_c}\right) \\
&= (iD_F)_{xa}(iD_F)_{yb}(iD_F)_{zc}\frac{\delta^3\Gamma}{\delta\varphi_a\delta\varphi_b\delta\varphi_c} \ . \qquad (41)
\end{aligned}$$

Rewriting the same derivative as

$$\frac{\delta}{\delta J_x}(iD_F)_{yz} = \frac{\delta}{\delta J_x}\left(\frac{-\delta^2\Gamma}{\delta\varphi_x\delta\varphi_a}\right)^{-1} = (iD_F)_{xa}(iD_F)_{yb}(iD_F)_{zc}\frac{\delta^3\Gamma}{\delta\varphi_a\delta\varphi_b\delta\varphi_c} \ . \qquad (42)$$

we get for the four point function $\frac{\delta^4 W}{\delta J_w \delta J_x \delta J_y \delta J_z}$

$$\begin{aligned}
&= \frac{\delta}{\delta J_w}\frac{\delta^3 W}{\delta J_x \delta J_y \delta J_z} = \frac{\delta}{\delta J_w}\left[(iD_F)_{xa}(iD_F)_{yb}(iD_F)_{zc}\frac{\delta^3\Gamma}{\delta\varphi_a\delta\varphi_b\delta\varphi_c}\right] \\
&= \left[(iD_F)_{wd}(iD_F)_{xe}(iD_F)_{af}\frac{\delta^3\Gamma}{\delta\varphi_d\delta\varphi_e\delta\varphi_f}\right](iD_F)_{yb}(iD_F)_{zc}\frac{\delta^3\Gamma}{\delta\varphi_a\delta\varphi_b\delta\varphi_c} \\
&\quad +(iD_F)_{xa}\left[(iD_F)_{wd}(iD_F)_{ye}(iD_F)_{bf}\frac{\delta^3\Gamma}{\delta\varphi_d\delta\varphi_e\delta\varphi_f}\right](iD_F)_{zc}\frac{\delta^3\Gamma}{\delta\varphi_a\delta\varphi_b\delta\varphi_c} \\
&\quad +(iD_F)_{xa}(iD_F)_{yb}\left[(iD_F)_{wd}(iD_F)_{ze}(iD_F)_{cf}\frac{\delta^3\Gamma}{\delta\varphi_d\delta\varphi_e\delta\varphi_f}\right]\frac{\delta^3\Gamma}{\delta\varphi_a\delta\varphi_b\delta\varphi_c} \\
&\quad +(iD_F)_{xa}(iD_F)_{yb}(iD_F)_{zc}(iD_F)_{wd}\frac{\delta^4\Gamma}{\delta\varphi_a\delta\varphi_b\delta\varphi_c\delta\varphi_d} \ ,
\end{aligned} \tag{43}$$

where the brackets $[\ldots]$ indicate the derivatives of the propagators (iD_F). This equation already shows, that the connected diagrams can be constructed by glueing vertices together with propagators.

And since the propagators are only the inverse of the second derivative of Γ, we need only calculate the derivatives of Γ to the required accuracy to obtain all information about the investigated process.

5 ABC-Theory example

So let us calculate now $\frac{\delta^n\Gamma}{(\delta\varphi)^n}$ for the ABC-theory. Here we have to go back and express everything in terms of the generating functional Z, since this is the only quantity that we can calculate directly in perturbation theory. We will use the form

$$Z(J_A, J_B, J_C; g) = \exp\left\{-g\int_z \frac{\delta^3}{\delta J_{Az}\,\delta J_{Bz}\,\delta J_{Cz}}\right\} \times Z(J_A, J_B, J_C; 0) \ . \tag{44}$$

We have to treat the classical field A as depending on its source J_A. The quantum fields do not appear anymore, as they are integrated out and the sources are independent from each other:

$$\frac{\delta J_j(x)}{\delta J_k(y)} = \delta^j_k \delta(x-y) \ , \quad \text{where } j,k \in \{A,B,C\} \ . \tag{45}$$

In the same way, the propagator is assumed to be independent of the sources:

$$\frac{\delta(iD_{Fj})_{xy}}{\delta J_k(z)} = \frac{\delta(iD^{-1}_{Fj})_{xy}}{\delta J_k(z)} = 0 \ , \quad \text{where } j,k \in \{A,B,C\} \ , \tag{46}$$

although the propagator is the second derivative of W or the inverse of the second derivative of Γ. These relations tell, how to expand the full propagator into subdiagrams.

The first derivative

$$\begin{aligned}
\frac{\delta\Gamma}{\delta A_x} &= \frac{\delta}{\delta A_x}[W - J_{Az}A_z - J_{Bz}B_z - J_{Cz}C_z] \\
&= (iD_{FA}^{-1})_{xy}\frac{\delta W}{\delta J_{Ay}} - J_{Az}\delta(x-z) - (iD_{FA}^{-1})_{xy}\frac{\delta J_{Az}}{\delta J_{Ay}}A_z - (iD_{FA}^{-1})_{xy}\frac{\delta}{\delta J_{Ay}}[J_{Bz}B_z + J_{Cz}C_z] \\
&= -J_{Ax} + (iD_{FA}^{-1})_{xy}[\frac{\delta W}{\delta J_{Ay}} - \delta(y-z)A_z - J_{Bz}\frac{\delta B_z}{\delta J_{Ay}} - J_{Cz}\frac{\delta C_z}{\delta J_{Ay}}] \qquad (47)
\end{aligned}$$

is proportional to sources when the definition of the classical fields $\varphi = \frac{\delta W}{\delta J}$ are plugged in:

$$\begin{aligned}
\frac{\delta\Gamma}{\delta A_x} &= -J_{Ax} + (iD_{FA}^{-1})_{xy}[\frac{\delta W}{\delta J_{Ay}} - \delta(y-z)\frac{\delta W}{\delta J_{Az}} - J_{Bz}\frac{\delta}{\delta J_{Ay}}\frac{\delta W}{\delta J_{Bz}} - J_{Cz}\frac{\delta}{\delta J_{Ay}}\frac{\delta W}{\delta J_{Cz}}] \\
&= -J_{Ax} - (iD_{FA}^{-1})_{xy}[J_{Bz}\frac{\delta}{\delta J_{Ay}}(\frac{-i}{Z}\frac{\delta Z}{\delta J_{Bz}}) + J_{Cz}\frac{\delta}{\delta J_{Ay}}(\frac{-i}{Z}\frac{\delta Z}{\delta J_{Cz}})] \\
&= 0 \qquad \text{for} \quad J_\phi = 0 \ . \qquad (48)
\end{aligned}$$

We get a similar result for the other fields B and C:

$$\left.\frac{\delta\Gamma}{\delta A_x}\right|_{J_\phi=0} = \left.\frac{\delta\Gamma}{\delta B_x}\right|_{J_\phi=0} = \left.\frac{\delta\Gamma}{\delta C_x}\right|_{J_\phi=0} = 0 \ . \qquad (49)$$

I kept here the formal derivation and did not do the normal expansion into powers of the coupling, as this first derivative can be done in a short enough way. I will still discuss the second and third derivative on general grounds in order to show, that only the higher derivatives of the generating functional have to be considered for the Feynman rules.

From the second derivatives

$$\begin{aligned}
\frac{\delta^2\Gamma}{\delta B_y\delta A_x} &= \frac{\delta^2}{\delta B_y\delta A_x}[W - J_{Az}A_z - J_{Bz}B_z - J_{Cz}C_z] \\
&= \frac{\delta^2 W}{\delta B_y\delta A_x} - \frac{\delta^2 J_{Az}}{\delta B_y\delta A_x}A_z - J_{Az}\frac{\delta^2 A_z}{\delta B_y\delta A_x} - \frac{\delta^2(J_{Bz}B_z)}{\delta B_y\delta A_x} - \frac{\delta^2(J_{Cz}C_z)}{\delta B_y\delta A_x} \qquad (50)
\end{aligned}$$

we see, that

$$\frac{\delta^2 J_{Az}}{\delta B_y\delta A_x}A_z = \frac{\delta}{\delta B_y}[(iD_{FA}^{-1})_{xw}\frac{\delta J_{Az}}{\delta J_{Aw}}]A_z = \frac{\delta}{\delta B_y}[(iD_{FA}^{-1})_{xw}\delta(z-w)]A_z = 0 \qquad (51)$$

and

$$J_{Az}\frac{\delta^2 A_z}{\delta B_y\delta A_x} = J_{Az}\frac{\delta}{\delta B_y}[\delta(z-x)] = 0 \qquad (52)$$

and similar for the other parts, so that only the derivative of W remains:

$$\begin{aligned}
\frac{\delta^2\Gamma}{\delta B_y\delta A_x} &= (iD_{FA}^{-1})_{xu}(iD_{FB}^{-1})_{yw}\frac{\delta^2 W}{\delta J_{Bw}\delta J_{Au}} = -i(iD_{FA}^{-1})_{xu}(iD_{FB}^{-1})_{yw}\frac{\delta^2 \ln Z}{\delta J_{Bw}\delta J_{Au}} \\
&= -i(iD_{FA}^{-1})_{xu}(iD_{FB}^{-1})_{yw}\frac{\delta}{\delta J_{Bw}}[\frac{1}{Z}\frac{\delta Z}{\delta J_{Au}}] \\
&= -i(iD_{FA}^{-1})_{xu}(iD_{FB}^{-1})_{yw}[\frac{1}{Z}\frac{\delta^2 Z}{\delta J_{Bw}\delta J_{Au}} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_{Bw}}\frac{\delta Z}{\delta J_{Au}}] \qquad (53)
\end{aligned}$$

and

$$\frac{\delta^2\Gamma}{\delta A_y \delta A_x} = -i(iD_{FA}^{-1})_{xu}(iD_{FA}^{-1})_{yw}[\frac{1}{Z}\frac{\delta^2 Z}{\delta J_{Aw}\delta J_{Au}} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_{Aw}}\frac{\delta Z}{\delta J_{Au}}] \ . \tag{54}$$

Using

$$Z = Ne^{-g\int_z \frac{\delta^3}{\delta J_{Az}\,\delta J_{Bz}\,\delta J_{Cz}}} \times e^{\frac{i}{2}J_{Ax}(iD_{FA})_{xy}J_{Ay}+\frac{i}{2}J_{Bx}(iD_{FB})_{xy}J_{By}+\frac{i}{2}J_{Cx}(iD_{FC})_{xy}J_{Cy}} \tag{55}$$

we get for $\frac{\delta Z}{\delta J_{Au}}$ always a source down from the exponent, that stays and is set to zero at the end. The contribution to $\mathcal{O}(g^k)$ gives

$$\frac{\delta Z}{\delta J_{Au}} \propto \sum_n J_A^{2n-k-1} J_B^{2n-k} J_C^{2n-k} \ . \tag{56}$$

There are no integers k and n such that

$$2n-k-1=0 \qquad \textbf{and} \qquad 2n-k=0 \tag{57}$$

and hence

$$\left.\frac{\delta Z}{\delta J_{Au}}\right|_{J=0} = 0 \ . \tag{58}$$

A similar analysis holds for $\frac{\delta^2 Z}{\delta J_{Bw}\delta J_{Au}}$. The contribution to $\mathcal{O}(g^k)$ gives

$$\frac{\delta^2 Z}{\delta J_{Bw}\delta J_{Au}} \propto \sum_n J_A^{2n-k-1} J_B^{2n-k-1} J_C^{2n-k} \ , \tag{59}$$

which gives the same equations as before and hence

$$\left.\frac{\delta^2 Z}{\delta J_{Bw}\delta J_{Au}}\right|_{J=0} = 0 \ . \tag{60}$$

But for $\frac{\delta^2 Z}{\delta J_{Aw}\delta J_{Au}}$ we get the contribution to $\mathcal{O}(g^k)$

$$\frac{\delta^2 Z}{\delta J_{Aw}\delta J_{Au}} \propto \sum_n J_A^{2n-k-2} J_B^{2n-k} J_C^{2n-k} \ , \tag{61}$$

which has vanishing exponents, and hence a finite value, for $k = 2n+2$. So the lowest order for a one-particle-irreducible (1PI) two-point function is $k = 2$:

$$\begin{aligned}\left.\frac{\delta^2\Gamma^{[2]}}{\delta A_y \delta A_x}\right|_{J=0} &= -i(iD_{FA}^{-1})_{xu}(iD_{FA}^{-1})_{yw}\left[\frac{1}{Z}\frac{\delta^2 Z}{\delta J_{Aw}\delta J_{Au}} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_{Aw}}\frac{\delta Z}{\delta J_{Au}}\right]_{J=0,\ \mathcal{O}(g^2)} \\ &= -i(iD_{FA}^{-1})_{xu}(iD_{FA}^{-1})_{yw}\left[Z(0;0)+Z(0;g^2)\right]^{-1} \\ &\quad \times \left[\frac{\delta^2}{\delta J_{Aw}\delta J_{Au}}(Z(J;0)+Z(J;g^2))\right]_{J=0,\ \mathcal{O}(g^2)}\end{aligned} \tag{62}$$

where

$$
\begin{aligned}
Z(J;0) &= N \times e^{\frac{i}{2}J_{Aa}(iD_{FA})_{ab}J_{Ab}+\frac{i}{2}J_{Ba}(iD_{FB})_{ab}J_{Bb}+\frac{i}{2}J_{Ca}(iD_{FC})_{ab}J_{Cb}} \\
Z(J;g^2) &= \frac{1}{2!}(-g)^2 \int_1 \frac{\delta^3}{\delta J_{A1}\,\delta J_{B1}\,\delta J_{C1}} \int_2 \frac{\delta^3}{\delta J_{A2}\,\delta J_{B2}\,\delta J_{C2}} \times Z(J;0) \qquad (63)
\end{aligned}
$$

and hence $Z(0;0) = N$ and

$$
\begin{aligned}
Z(0;g^2) &= \frac{1}{2!}(-g)^2 \int_1 \frac{\delta^3}{\delta J_{A1}\,\delta J_{B1}\,\delta J_{C1}} \int_2 \frac{\delta^3}{\delta J_{A2}\,\delta J_{B2}\,\delta J_{C2}} \times Z(J;0)\bigg|_{J=0} \\
&= -N\frac{g^2}{2}\int_{1,2}(D_{FA})_{12}(D_{FB})_{12}(D_{FC})_{12} \ . \qquad (64)
\end{aligned}
$$

Further

$$
\left[\frac{\delta^2}{\delta J_{Aw}\delta J_{Au}} Z(J;0)\right]_{J=0} = Ni(iD_{FA})_{wu} = -N(D_{FA})_{wu} \qquad (65)
$$

and

$$
\begin{aligned}
&\left[\frac{\delta^2}{\delta J_{Aw}\delta J_{Au}} Z(J;g^2)\right]_{J=0} \\
&= \frac{\delta^2}{\delta J_{Aw}\delta J_{Au}}\frac{g^2}{2}\int_1 \frac{\delta^3}{\delta J_{A1}\,\delta J_{B1}\,\delta J_{C1}} \int_2 \frac{\delta^3}{\delta J_{A2}\,\delta J_{B2}\,\delta J_{C2}} \times Z(J;0)\bigg|_{J=0} \\
&= N\frac{g^2}{2}(-1)\int_{1,2}(iD_{FB})_{12}(iD_{FC})_{12}\frac{\delta^4}{\delta J_{Aw}\delta J_{Au}\delta J_{A1}\delta J_{A2}}\frac{1}{2!}(\tfrac{i}{2}J_{Aa}(iD_{FA})_{ab}J_{Ab})^2 \qquad (66) \\
&= N\frac{g^2}{2}\int_{1,2}(D_{FB})_{12}(D_{FC})_{12}\left[(D_{FA})_{wu}(D_{FA})_{12} + (D_{FA})_{w1}(D_{FA})_{u2} + (D_{FA})_{w2}(D_{FA})_{u1}\right] \ .
\end{aligned}
$$

So

$$
\begin{aligned}
&\frac{\delta^2\Gamma^{[0+2]}}{\delta A_y \delta A_x}\bigg|_{J=0} \\
&= i(D_{FA}^{-1})_{xu}(D_{FA}^{-1})_{yw}\left[N - N\frac{g^2}{2}\int_{1,2}(D_{FA})_{12}(D_{FB})_{12}(D_{FC})_{12}\right]^{-1} \\
&\quad \times\Big[-N(D_{FA})_{wu} + N\frac{g^2}{2}\int_{1,2}(D_{FB})_{12}(D_{FC})_{12}[(D_{FA})_{wu}(D_{FA})_{12} \\
&\qquad\qquad +(D_{FA})_{w1}(D_{FA})_{u2} + (D_{FA})_{w2}(D_{FA})_{u1}]\Big] \\
&= i(D_{FA}^{-1})_{xu}(D_{FA}^{-1})_{yw}(-(D_{FA})_{wu}) \\
&\quad +i(D_{FA}^{-1})_{xu}(D_{FA}^{-1})_{yw}\frac{g^2}{2}\int_{1,2}(D_{FB})_{12}(D_{FC})_{12}[(D_{FA})_{w1}(D_{FA})_{u2} + (D_{FA})_{w2}(D_{FA})_{u1}] \\
&= -i(D_{FA}^{-1})_{xu}\delta(y-u) \\
&\quad +i\frac{g^2}{2}\int_{1,2}(D_{FB})_{12}(D_{FC})_{12}[\delta(y-1)\delta(2-x) + \delta(y-2)\delta(1-x)] \\
&= -(iD_{FA}^{-1})_{xy} + ig^2\frac{1}{2}[(D_{FB})_{yx}(D_{FC})_{yx} + (D_{FB})_{xy}(D_{FC})_{xy}] \\
&= -(iD_{FA}^{-1})_{xy} - ig^2(iD_{FB})_{xy}(iD_{FC})_{xy} \ , \qquad (67)
\end{aligned}
$$

where the first term is the definition of the propagator and the second term is the 1PI two-point function to order g^2, which is just two other propagators connecting the two points.

With a similar argumentation we find, that the only non-vanishing three-point function is

$$\frac{\delta^3\Gamma}{\delta C_z \delta B_y \delta A_x} = (iD_{FA}^{-1})_{xu}(iD_{FB}^{-1})_{yw}(iD_{FC}^{-1})_{zt}\frac{\delta^3 W}{\delta J_{Ct}\delta J_{Bw}\delta J_{Au}} \tag{68}$$

and we again only need to investigate

$$\frac{\delta^3 Z}{\delta J_{Ct}\delta J_{Bw}\delta J_{Au}} \propto \sum_n J_A^{2n-k-1} J_B^{2n-k-1} J_C^{2n-k-1} \ , \tag{69}$$

which has solutions for $k = 2n+1$ with the lowest order $k = 1$. The calculation is even simpler, since

$$Z(0, g^1) = (-g)\int_1 \frac{\delta^3}{\delta J_{A1}\,\delta J_{B1}\,\delta J_{C1}} \times Z(J;0)\bigg|_{J=0} = 0 \ . \tag{70}$$

Now

$$\begin{aligned} \frac{\delta^3 Z^{[1]}}{\delta J_{Ct}\delta J_{Bw}\delta J_{Au}} &= N\,\frac{\delta^3}{\delta J_{Ct}\delta J_{Bw}\delta J_{Au}}(-g)\int_1 \frac{\delta^3}{\delta J_{A1}\,\delta J_{B1}\,\delta J_{C1}} \times Z(J;0)\bigg|_{J=0} \\ &= -Ng(-i)\int_1 (iD_{FA})_{u1}(iD_{FB})_{w1}(iD_{FC})_{t1} \end{aligned} \tag{71}$$

and so

$$\begin{aligned} \frac{\delta^3 \Gamma^{[1]}}{\delta C_z \delta B_y \delta A_x} &= (iD_{FA}^{-1})_{xu}(iD_{FB}^{-1})_{yw}(iD_{FC}^{-1})_{zt}[N]^{-1} \times Ng\int_1 (D_{FA})_{u1}(D_{FB})_{w1}(D_{FC})_{t1} \\ &= -ig\int_1 \delta(x-1)\delta(y-1)\delta(z-1) \ . \end{aligned} \tag{72}$$

If we want to write now the Feynman rules not in position space but momentum space, we have for each field the plane-wave factor e^{ipx}. Writing these factors down, by writing the propagators with their Fourier transformations, we can integrate over the arbitrary point 1 and obtain

$$\begin{aligned} &\Gamma[q_A, q_B, q_C; g^1] \\ &= \int d^4x e^{iq_A x}\int d^4y e^{iq_B y}\int d^4z e^{iq_C z}\frac{\delta^3\Gamma^{[1]}}{\delta C_z \delta B_y \delta A_x} \\ &= -ig\int d^4x e^{iq_A x}\int d^4y e^{iq_B y}\int d^4z e^{iq_C z}\int d^4t\int d^4u\int d^4w\int d^4 1 \\ &\qquad \int \frac{d^4p_A}{(2\pi)^4} i e^{-ip_A(x-u)}(-p_A^2+m_A^2) \\ &\qquad \int \frac{d^4p_B}{(2\pi)^4} i e^{-ip_B(y-w)}(-p_B^2+m_B^2)\int \frac{d^4p_C}{(2\pi)^4} i e^{-ip_C(z-t)}(-p_C^2+m_C^2) \\ &\qquad \int \frac{d^4k_A}{(2\pi)^4}\frac{ie^{-ik_A(u-1)}}{k_A^2-m_A^2+i\epsilon}\int \frac{d^4k_B}{(2\pi)^4}\frac{ie^{-ik_B(w-1)}}{k_B^2-m_B^2+i\epsilon}\int \frac{d^4k_C}{(2\pi)^4}\frac{ie^{-ik_C(t-1)}}{k_C^2-m_C^2+i\epsilon} \end{aligned} \tag{73}$$

and further

$$\begin{aligned}
&\Gamma[q_A, q_B, q_C; g^1] \\
&= -ig \int \frac{d^4p_A}{(2\pi)^4}(p_A^2 - m_A^2) \int \frac{d^4p_B}{(2\pi)^4}(p_B^2 - m_B^2) \int \frac{d^4p_C}{(2\pi)^4}(p_C^2 - m_C^2) \\
&\quad \int \frac{d^4k_A}{(2\pi)^4} \frac{1}{k_A^2 - m_A^2 + i\epsilon} \int \frac{d^4k_B}{(2\pi)^4} \frac{1}{k_B^2 - m_B^2 + i\epsilon} \int \frac{d^4k_C}{(2\pi)^4} \frac{1}{k_C^2 - m_C^2 + i\epsilon} \\
&\quad \int d^4x e^{i(q_A - p_A)x} \int d^4y e^{i(q_B - p_B)y} \int d^4z e^{i(q_C - p_C)z} \int d^4t \int d^4u \int d^4w \int d^41 \\
&\quad e^{i(p_A - k_A)u} e^{i(p_B - k_B)w} e^{i(p_C - k_C)t} e^{i(k_A + k_B + k_C)1} \\
&= -ig \int \frac{d^4p_A}{(2\pi)^4}(p_A^2 - m_A^2) \int \frac{d^4p_B}{(2\pi)^4}(p_B^2 - m_B^2) \int \frac{d^4p_C}{(2\pi)^4}(p_C^2 - m_C^2) \\
&\quad \int \frac{d^4k_A}{(2\pi)^4} \frac{1}{k_A^2 - m_A^2 + i\epsilon} \int \frac{d^4k_B}{(2\pi)^4} \frac{1}{k_B^2 - m_B^2 + i\epsilon} \int \frac{d^4k_C}{(2\pi)^4} \frac{1}{k_C^2 - m_C^2 + i\epsilon} \\
&\quad (2\pi)^4\delta(q_A - p_A)(2\pi)^4\delta(q_B - p_B)(2\pi)^4\delta(q_C - p_C) \\
&\quad (2\pi)^4\delta(p_A - k_A)(2\pi)^4\delta(p_B - k_B)(2\pi)^4\delta(p_C - k_C)(2\pi)^4\delta(k_A + k_B + k_C) \\
&= -ig \int \frac{d^4p_A}{(2\pi)^4} \frac{(p_A^2 - m_A^2)}{p_A^2 - m_A^2 + i\epsilon} \int \frac{d^4p_B}{(2\pi)^4} \frac{(p_B^2 - m_B^2)}{p_B^2 - m_B^2 + i\epsilon} \int \frac{d^4p_C}{(2\pi)^4} \frac{(p_C^2 - m_C^2)}{p_C^2 - m_C^2 + i\epsilon} \\
&\quad (2\pi)^4\delta(q_A - p_A)(2\pi)^4\delta(q_B - p_B)(2\pi)^4\delta(q_C - p_C)(2\pi)^4\delta(p_A + p_B + p_C) \\
&= -ig(2\pi)^4\delta(q_A + q_B + q_C) \ , \qquad (74)
\end{aligned}$$

which is just the Feynman rule for the simple vertex in momentum space: vertex factor times momentum conservation.

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