3. Special Relativity (SR) — Examples

Example 1: going into a restframe by a Lorentz transformation (LT)

- we see a particle (or a train, or . . .) moving with the speed v in $\hat{x}$ direction
  - for simplicity we suppress $\hat{y}$ and $\hat{z}$ directions $\Rightarrow$ only two component vectors
  - the velocity between our frame O and the particles restframe O' is $\vec{v} = v\,\hat{x}$
  - the LT $O \to O'$ is given by $\vec{\beta} = \frac{\vec{v}}{c}$: $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$
  - the LT $O' \to O$ is given by $\vec{\beta}' = \frac{\vec{v}'}{c} = -\frac{\vec{v}}{c}$: $\Lambda' = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$
  - the four momentum of the particle in O' (its restframe !) is $p'^{\mu} = \begin{pmatrix} m \\ \vec{0} \end{pmatrix}$
    - its mass m is the energy of the four momentum in its restframe
  - the four momentum in O is $p^{\mu} = \Lambda'^{\mu}{}_{\nu} p'^{\nu}$ or

$$p^{\mu} = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\vec{\beta} \\ \gamma\vec{\beta} & \gamma \end{pmatrix} \cdot \begin{pmatrix} m \\ \vec{0} \end{pmatrix} = \begin{pmatrix} m\gamma \\ m\gamma\vec{\beta} \end{pmatrix} \Rightarrow \begin{cases} E = m\gamma \\ \vec{p} = m\gamma\vec{\beta} \end{cases}$$

- solving these equations for the velocity $\vec{v} = \vec{\beta}c$ gives $\vec{v} = \frac{\vec{p}}{E}c$

3. Special Relativity (SR) — Examples

Example 2: energy release in a fixed target experiment

- ''6 GeV protons'' (p_B^μ) from the Bevatron hit stationary protons (p_A^μ) in the detector
  - for simplicity we suppress $\hat{y}$ and $\hat{z}$ directions $\Rightarrow$ only two component vectors
- in the Lab frame we know all initial quantities:
  - the four momentum of the stationary protons: $p_A^\mu = \begin{pmatrix} m_p \\ \vec{0} \end{pmatrix}$
  - the kinetic energy of the accelerated protons $E_B - m_p = 6.5\,\text{GeV}$
  - through the energy-momentum-mass relation:
    * the momentum of the accelerated protons $p_B = \sqrt{E_B^2 - m_p^2}$
  - the total four momentum in the Lab frame is:

$$P^\mu = p_A^\mu + p_B^\mu = \begin{pmatrix} E_B + m_p \\ \vec{p}_B \end{pmatrix}$$

- in the CM frame of the reaction we see the available energy:
  - the momentum of the reaction products is zero $\quad Q^\mu = \begin{pmatrix} E_{CM} \\ \vec{0} \end{pmatrix}$

Example 2: energy release in a fixed target experiment ... continued

- How can we connect Lab frame and CM frame ?
  - by a Lorentz transformation (like example 1)
  - by noting that there is an invariant
    - something that is the same in both frames

- the simplest invariant is the ”square” of the total four momentum

$$
\begin{aligned}
P^2 &= P^\mu P_\mu = (m_p + E_B)^2 - (\vec{p}_B)^2 = {m_p}^2 + 2m_p E_B + {E_B}^2 - (\vec{p}_B)^2 \\
&= {m_p}^2 + 2m_p E_B + {m_p}^2 = 2m_p(m_p + E_B)
\end{aligned}
$$

and

$$Q^2 = Q^\mu Q_\mu = (E_{CM})^2 - (\vec{0})^2 = E_{CM}^2$$

- this give the available energy as $E_{CM} = \sqrt{2m_p(m_p + E_B)}$
  - with $m_p = 938\,\text{MeV}$ and $E_B = 7438\,\text{MeV}$ we get $E_{CM} \approx 3964\,\text{MeV} > 4m_p$
  - the Bevatron energy made the reaction possible: $p + p \to p + p + p + \bar{p}$

3. Special Relativity (SR) — Algebra of the Poincaré group (optional)

Groups — what is a group? (repetition)

- a set G together with a "multiplication $\circ$" with the properties:
  - for $a, b \in G \Rightarrow c = a \circ b \in G$
  - $(a \circ b) \circ c = a \circ (b \circ c)$
  - $\forall a \in G : \exists e \in G$ with $a \circ e = e \circ a = a$
  - $\forall a \in G : \exists a^{-1} \in G$ with $a \circ a^{-1} = a^{-1} \circ a = e$
- if $a \circ b = b \circ a \; \forall a, b \in G$: abelian group, otherwise non-abelian
  - abelian: $\{\mathcal{R}, +\}$ or $\{\mathcal{R}^+, \times\}$
  - non-abelian: regular square matrices with the matrix multiplication
- continuous groups: the elements depend on a continuous parameter
  - example: rotations around an axis $R[\theta]$ with $\theta \in [0, 2\pi)$
- Lie group: a continuous group with an analytic multiplication
  - $g[\vec{x}] \circ g[\vec{y}] = g[f(\vec{x}, \vec{y})]$ with $f(\vec{x}, \vec{y})$ analytic in $\vec{x}$ and $\vec{y}$
  - the unit element is $e = g[\vec{0}]$

Lie groups and Lie algebras

- The $n \times n$ (complex) matrices form representations of Lie groups

- group multiplication is analytic $\Rightarrow$ expansion around unit element
  - unit element $e = \mathbf{1}_{n\times n}$
  - representation $T(g[\alpha]) = \exp[i\alpha_i X_i]$ $\Rightarrow$ $X_k = -i\frac{\partial T(g[\alpha])}{\partial \alpha_k}|_{\vec{\alpha}=0}$
  - generators $\{X_k\}$ span the representation of the Lie group

- the generators $\{X_k\}$ fulfill the Lie algebra $[X_j, X_k] = C_{jk}^{\ \ \ell} X_\ell$
  - with the antisymmetric structure constants $C_{jk}^{\ \ \ell} = -C_{kj}^{\ \ \ell}$
  - rank of the group: number of commuting generators
  - a Casimir operator commutes with all generators $\Rightarrow$ $\propto e$

- the indices i, j, k, ℓ need not indicate single numbers!
  - for the generators we will have $X_i = X_{[mn]} = -X_{[nm]}$

Representations of the Lie group

- using the Jacobi identity

$$
\begin{aligned}
0 &= [A,[B,C]] + [B,[C,A]] + [C,[A,B]] \\
&= ABC - ACB - BCA + CBA \;+\; BCA - BAC - CAB + ACB \;+\; CAB - CBA - ABC + BAC
\end{aligned}
$$

we get for the structure constants

$$
\begin{aligned}
0 &= C_{bc}{}^{d}[A,D] + C_{ca}{}^{d}[B,D] + C_{ab}{}^{d}[C,D] \\
&= C_{bc}{}^{d}C_{ad}{}^{e} + C_{ca}{}^{d}C_{bd}{}^{e} + C_{ab}{}^{d}C_{cd}{}^{e} = -(C_{ca}{}^{d}C_{db}{}^{e} - C_{cb}{}^{d}C_{da}{}^{e}) + C_{ab}{}^{d}C_{cd}{}^{e}
\end{aligned}
$$

- writing the structure constants as matrices $(M_k)_j{}^{\ell} = C_{jk}{}^{\ell}$ we have

$$
0 = -[(M_a)_c{}^{d}(M_b)_d{}^{e} - (M_b)_c{}^{d}(M_a)_d{}^{e}] + C_{ab}{}^{d}(M_d)_c{}^{e}
$$

or

$$
[M_a, M_b] = C_{ab}{}^{d} M_d
$$

$\Rightarrow$ structure constants form the adjoint representation of the Lie group

Lie Algebra of the rotation group

- a rotation around the $\hat{z}$-axis by the angle θ is done by the matrix

$$R[\theta] \;=\; \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{i\theta L_z}$$

  – in index notation: $R[\theta]^j{}_k = \cos\theta(\delta^j_1\delta^1_k + \delta^j_2\delta^2_k) - \sin\theta(\delta^j_1\delta^2_k - \delta^j_2\delta^1_k) + \delta^j_3\delta^3_k$

- so the generator of the rotations, iL_z, is

$$iL_z = \left.\frac{\partial R[\theta]}{\partial\theta}\right|_{\theta=0} = \left.\begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\right|_{\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  – and similar

$$iL_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad iL_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- these rotations (incl. L_x and L_y) act on 3d column vectors $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

Lie Algebra of the rotation group

- with simple matrix multiplication we can see:

$$[iL_x, iL_y] = -iL_z \qquad [iL_y, iL_z] = -iL_x \qquad [iL_z, iL_x] = -iL_y$$

  - or in index notation with $x = 1$, $y = 2$, and $z = 3$: $[L_j, L_k] = i\epsilon_{jk\ell}L_\ell$

- but there is a smaller dimensional realisation of the rotation group!
  - using the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  - one can define the Spin matrices $S_k = \frac{1}{2}\sigma_k$, which give

$$[S_j, S_k] = i\epsilon_{jk\ell}S_\ell$$

- these Spin matrices act on 2d complex column vectors $\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$ $\Rightarrow$ Spinors

$\Rightarrow$ fundamental representation of the rotation group $SU(2)$

Rotations of Spinors

- with simple matrix multiplication we can see for the Pauli matrices:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{2\times 2}$$

- So the finite rotation of a spinor around the $\hat{y}$-axis is

$$\begin{aligned} R[\theta] &= e^{i\theta S_y} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta \tfrac{1}{2}\sigma_y)^n = \sum_{n=2m} \frac{1}{n!}(i\tfrac{\theta}{2})^n \sigma_y^n + \sum_{n=2m+1} \frac{1}{n!}(i\tfrac{\theta}{2})^n \sigma_y^n \\ &= \sum_n^{\infty} \frac{(-1)^n(\frac{\theta}{2})^{2n}}{(2n)!}(\sigma_y^2)^n + i\sum_n^{\infty} \frac{(-1)^n(\frac{\theta}{2})^{2n+1}}{(2n+1)!}(\sigma_y^2)^n \sigma_y \\ &= \cos\tfrac{\theta}{2} * 1_{2\times 2} + i\sin\tfrac{\theta}{2}\sigma_y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \end{aligned}$$

  - acting on the spinor $\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\Rightarrow$ spinors rotate only with half of the rotation angle θ

Lorentz transformations (like Galilean transformations) consist of Boosts and Rotations

- a boost in $\hat{x}$ was done by

$$\Lambda(\eta)^\mu{}_\nu = \cosh\eta(\delta^\mu_0\delta^0_\nu + \delta^\mu_1\delta^1_\nu) - \sinh\eta(\delta^\mu_0\delta^1_\nu + \delta^\mu_1\delta^0_\nu) + \delta^\mu_2\delta^2_\nu + \delta^\mu_3\delta^3_\nu$$

- a rotation between $\hat{y}$ and $\hat{z}$ can be done by

$$\Lambda(\theta)^\mu{}_\nu = \delta^\mu_0\delta^0_\nu + \delta^\mu_1\delta^1_\nu + \cos\theta(\delta^\mu_2\delta^2_\nu + \delta^\mu_3\delta^3_\nu) - \sin\theta(\delta^\mu_2\delta^3_\nu - \delta^\mu_3\delta^2_\nu)$$

- we obtain the generators for boosts with $-i\frac{\partial\Lambda(\eta)^\mu{}_\nu}{\partial\eta}|_{\eta=0} =$

$$-i\sinh\eta(\delta^\mu_0\delta^0_\nu + \delta^\mu_1\delta^1_\nu) + i\cosh\eta(\delta^\mu_0\delta^1_\nu + \delta^\mu_1\delta^0_\nu)|_{\eta=0} = i(\delta^\mu_0\delta^1_\nu + \delta^\mu_1\delta^0_\nu)$$

- we obtain the generators for rotations with $-i\frac{\partial\Lambda(\theta)^\mu{}_\nu}{\partial\theta}|_{\theta=0} =$

$$+i\sin\theta(\delta^\mu_2\delta^2_\nu + \delta^\mu_3\delta^3_\nu) + i\cos\theta(\delta^\mu_2\delta^3_\nu - \delta^\mu_3\delta^2_\nu)|_{\theta=0} = i(\delta^\mu_2\delta^3_\nu - \delta^\mu_3\delta^2_\nu)$$

Boosts and Rotations ... continued

- The other boosts go in $\hat{y}$ or $\hat{z}$ direction: $i(\delta^\mu_0 \delta^i_\nu + \delta^\mu_i \delta^0_\nu)$,
  or with the indices $0i$ down: $(M_{0i})^\mu{}_\nu = i(\delta^\mu_0(-g_{i\nu}) + \delta^\mu_i g_{0\nu})$
- The other rotations go in $\hat{x}\hat{y}$ or $\hat{x}\hat{z}$ direction: $i(\delta^\mu_j \delta^k_\nu - \delta^\mu_k \delta^j_\nu)$,
  or with the indices jk down: $(M_{jk})^\mu{}_\nu = i(\delta^\mu_j(-g_{k\nu}) - \delta^\mu_k(-g_{j\nu}))$
- both generators have now the same form:

$$(M_{\alpha\beta})^\mu{}_\nu = -i(\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu})$$

- with $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ we get

$$\Lambda(\omega)^\mu{}_\nu = \exp[i(M_{\alpha\beta}\omega^{\alpha\beta})^\mu{}_\nu] = \exp[(\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu})\omega^{\alpha\beta}]$$

- How to understand / use this formula? ... How to get a matrix?
  1. pick the indices of $\omega^{\alpha\beta}$: ω^{0i} (ω^{jk}) for a boost (rotation) in $\hat{i}$- ($\hat{j}\hat{k}$-) direction
  2. write the matrix $\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu}$ with row-(column-) number μ (ν)
     * it will only have two non-zero entries
  3. squaring the matrix gives a diagonal matrix with only two equal entries
  4. the powerseries expansion gives you the expected boost / rotation

Generators for the Lorentz transformations

- these generators fulfill the Lie algebra of the Lorentz group:

$$[M_{\alpha\beta}, M_{\gamma\delta}]^{\mu}{}_{\nu} = i(g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\delta}M_{\beta\gamma} + g_{\beta\delta}M_{\alpha\gamma})^{\mu}{}_{\nu}$$

- unifying time and spatial translations $P_\mu = (H, P_i)$
- we get the rest of the Poincaré algebra:

$$[P_\mu, P_\nu] = 0 \qquad \text{and} \qquad [M_{\alpha\beta}, P_\mu] = i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)$$

- the generators of the Poincaré group are: P_μ and $M_{\alpha\beta}$
  - all rotations, boosts, and translations are elements of the Poincaré group

Invariants of the Poincaré group

- are objects that commute with all elements of the Poincaré group
  - it is enough to check if they commute with the generators . . .

Invariants of the Poincaré group

- obviously $[ab, c] = a[b, c] + [a, c]b = abc - acb + acb - cab = abc - cab$
- so $[P_\mu, P^2] = [P_\mu, P_\nu]P^\nu + P^\nu[P_\mu, P_\nu] = 0$
- and
$$\begin{aligned} [M_{\alpha\beta}, P^2] &= g^{\mu\nu}[M_{\alpha\beta}, P_\mu]P_\nu + g^{\mu\nu}P_\mu[M_{\alpha\beta}, P_\nu] \\ &= g^{\mu\nu}i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)P_\nu + g^{\mu\nu}P_\mu i(g_{\alpha\nu}P_\beta - g_{\beta\nu}P_\alpha) \\ &= -2i[P_\alpha, P_\beta] = 0 \ . \end{aligned}$$

$\Rightarrow$ $P^2 = m^2$ invariant is a consequence of the Poincaré algebra!

- Another invariant is W^2
  - with the Pauli-Lubanski vector $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}M_{\nu\rho}P_\lambda$

$$\begin{aligned} [P_\kappa, W^\mu] &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}([P_\kappa, M_{\nu\rho}]P_\lambda + M_{\nu\rho}[P_\kappa, P_\lambda]) \\ &= \tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}i(g_{\rho\kappa}P_\nu - g_{\nu\kappa}P_\rho)P_\lambda = 0 \quad \Rightarrow \quad [P_\kappa, W^2] = 0 \end{aligned}$$

  - $0 = [M_{\alpha\beta}, W^2]$ is true, but checking is too difficult ...

$\Rightarrow$ Particles can be characterised by the eigenvalues of P^2 and W^2

Eigenvalues of P^2 and W^2

- the spin vector W^μ is orthogonal to P_μ :

$$(P.W) = P^\mu \frac{1}{2}\epsilon_{\mu\nu\rho\lambda} M^{\nu\rho} P^\lambda = 0$$

- For a particle at rest: $P_\mu = (m, 0)$
  - $P^2 = m^2 \Rightarrow$ the eigenvalue of P^2 is m^2
  - $W_\mu = \frac{1}{2} m \epsilon_{\mu\nu\rho 0} M^{\nu\rho} = m(0, \vec{J})$
  - so $W^2 = m^2(0^2 - \vec{J}^2) = -m^2 \vec{J}^2 \to -m^2 s(s+1)$

  $\Rightarrow$ the eigenvalue of W^2 is $m^2 s(s+1)$

- For a massless particle $P_\mu = (\eta, \eta, 0, 0)$
  - we have $P^2 = (P.W) = W^2 = 0$

    $\Rightarrow$ the eigenvalues of P^2 and W^2 are 0
  - we can construct the operator $0 = \lambda^2 P^2 - 2\lambda(P.W) + W^2 = (\lambda P - W)^2$
    * where λ depends on the representation (i.e. the spin) of the particle
  - we get: $W^\mu = \lambda P^\mu$ with the helicity $\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$

$\Rightarrow$ Particles are characterised by mass and spin !

Investigating the Lorentz group

- distinguishing again boosts and rotations

$$K_i = M_{0i} = -M^{0i} \qquad \text{and} \qquad J_i = \tfrac{1}{2}\epsilon_{ijk}M^{jk} \ ,$$

the Lorentz algebra gives

$$[J_j, J_k] = i\epsilon_{jk\ell}J_\ell \ , \quad [K_j, K_k] = -i\epsilon_{jk\ell}J_\ell \ , \quad [J_j, K_k] = i\epsilon_{jk\ell}K_\ell$$

- defining

$$L_i = N_i = \tfrac{1}{2}(J_i + iK_i) \qquad \text{and} \qquad R_i = N_i^\dagger = \tfrac{1}{2}(J_i - iK_i)$$

one gets

$$[L_j, R_k] = 0 \ , \quad [L_j, L_k] = i\epsilon_{jk\ell}L_\ell \ , \quad [R_j, R_k] = i\epsilon_{jk\ell}R_\ell$$

$\Rightarrow$ the Lorentz algebra is similar to $SU(2)_L \otimes SU(2)_R$!

- it has two invariants: $L_iL_i = n(n+1)$ and $R_iR_i = m(m+1)$
  - the angular momentum is $J_i = L_i + R_i \quad \Rightarrow \quad$ spin $j = n + m$

classifying particles

- according to the eigenstates (n, m) of $SU(2)_L \otimes SU(2)_R$
  - $(0,0)$ is a scalar
  - $(\frac{1}{2},0)$ is the χ_α left-handed Weyl-spinor
  - $(0,\frac{1}{2})$ is the $\bar{\eta}^{\dot{\alpha}}$ right-handed Weyl-spinor
  - $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ is $\Psi = \binom{\chi_\alpha}{\bar{\eta}^{\dot{\alpha}}}$, the Dirac-spinor
  - $(\frac{1}{2},0) \otimes (0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$ is $(\chi\sigma^\mu\bar{\eta}) = \chi^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\eta}^{\dot{\alpha}}$, a spin-1 four-vector

$\Rightarrow$ in that sense is the spinor the square root of the vector

- under Parity: $J_i \xrightarrow{\mathsf{P}} J_i,\ K_i \xrightarrow{\mathsf{P}} -K_i, \quad \Rightarrow \quad L_i \xleftrightarrow{\mathsf{P}} R_i,\ (n,m) \xleftrightarrow{\mathsf{P}} (m,n)$
  - the scalar stays the same
  - $(\frac{1}{2},0) \xleftrightarrow{\mathsf{P}} (0,\frac{1}{2})$, therefore $\chi_\alpha \xleftrightarrow{\mathsf{P}} \bar{\eta}^{\dot{\alpha}}$
  - $(\frac{1}{2},0) \oplus (0,\frac{1}{2}) \xleftrightarrow{\mathsf{P}} (0,\frac{1}{2}) \oplus (\frac{1}{2},0) = (\frac{1}{2},0) \oplus (0,\frac{1}{2})$

    $\Rightarrow$ so a Dirac-spinor stays a Dirac-spinor
  - the four-vector stays the same