Special Relativity

David W. Hogg

School of Natural Sciences
Institute for Advanced Study
Olden Lane
Princeton NJ 08540

`hogg@ias.edu`

1 December 1997

Contents

Preface

For me, the wonder of special relativity lies in its successful prediction of interesting and very nonintuitive phenomena from simple arguments with simple premises.

These notes have three (perhaps ambitious) aims: (a) to introduce undergraduates to special relativity from its founding principle to its varied consequences, (b) to serve as a reference for those of us who need to use special relativity regularly but have no long-term memory, and (c) to provide an illustration of the methods of theoretical physics for which the elegance and simplicity of special relativity are ideally suited. History is a part of all science—I will mention some of the relevant events in the development of special relativity—but there is no attempt to present the material in a historical way.

A common confusion for students of special relativity is between that which is real and that which is apparent. For instance, length contraction is often mistakenly thought to be some optical illusion. But moving things do not "appear" shortened, they actually *are* shortened. How they appear depends on the particulars of the observation, including distance to the observer, viewing angles, times, etc. The observer finds that they are shortened only after correcting for these non-fundamental details of the observational procedure. I attempt to emphasize this distinction: All apparent effects, including the Doppler Shift, stellar aberration, and superluminal motion, are relegated to Chapter 7. I think these are very important aspects of special relativity, but from a pedagogical standpoint it is preferable to separate them from the basics, which are not dependent on the properties of the observer.

I love the description of special relativity in terms of frame-independent, geometric objects, such as scalars and 4-vectors. These are introduced in Chapter 6 and used thereafter. But even before this, the geometric properties of spacetime are emphasized. Most problems can be solved with a minimum of algebra; this is one of the many beautiful aspects of the subject.

These notes, first written while teaching sections of first-year physics at Caltech, truly represent a work in progress. I strongly encourage all readers to give me comments on any aspect of the text[*]; all input is greatly appreciated. Thank you very much.

Acknowledgments

David W. Hogg
Princeton, New Jersey
November 1997

---

[*]email: `hogg@ias.edu`

Chapter 1

Principles of relativity

These notes are devoted to the consequences of Einstein's (1905) principle of special relativity, which states that all the fundamental laws of physics are the same for all uniformly moving (non-accelerating) observers. In particular, all of them measure precisely the same value for the speed of light in vacuum, no matter what their relative velocities. Before Einstein wrote, several principles of relativity had been proposed, but Einstein was the first to state it clearly and hammer out all the counterintuitive consequences. In this Chapter the concept of a "principle of relativity" is introduced, Einstein's is presented, and some of the experimental evidence prompting it is discussed.

1.1 What is a principle of relativity?

The first principle of relativity ever proposed is attributed to Galileo, although he probably did not formulate it precisely. Galileo's principle of relativity says that sailors on a uniformly moving boat cannot, by performing on-board experiments, determine the boat's speed. They can determine the speed by looking at the relative movement of the shore, by dragging something in the water, or by measuring the strength of the wind, but there is no way they can determine it without observing the world outside the boat. A sailor locked in a windowless room cannot even tell whether the ship is sailing or docked*.

This is a principle of relativity, because it states that there are no observational consequences of absolute motion. One can only measure one's velocity *relative* to something else.

As physicists we are empiricists: we reject as meaningless any concept which has no observable consequences, so we conclude that there is no such thing as "absolute motion." Objects have velocities only with respect to one another. Any statement of an object's speed must be made with respect to something else.

Our language is misleading because we often give speeds with no reference object. For example, a police officer might say to you "Excuse me, but do you realize that you were driving at 85 miles per hour?" The officer

leaves out the phrase "with respect to the Earth," but it is there implicitly. In other words, you cannot contest a speeding ticket on the strength of Galileo's principle since it is implicit in the law that the speed is to be measured with respect to the road.

When Kepler first introduced a heliocentric model of the Solar System, it was resisted on the grounds of common sense. If the Earth is orbiting the Sun, why can't we "feel" the motion? Relativity provides the answer: there are no local, observational consequences to our motion.[†] Now that the Earth's motion is generally accepted, it has become the best evidence we have for Galilean relativity. On a day-to-day basis we are not aware of the motion of the Earth around the Sun, despite the fact that its orbital speed is a whopping 30 km s^{-1} ($100{,}000 \text{ km h}^{-1}$). We are also not aware of the Sun's 220 km s^{-1} motion around the center of the Galaxy (e.g., Binney & Tremaine 1987, Chapter 1) or the roughly 600 km s^{-1} motion of the local group of galaxies (which includes the Milky Way) relative to the rest frame of the cosmic background radiation (e.g., Peebles 1993, Section 6). We have become aware of these motions only by observing extraterrestrial references (in the above cases, the Sun, the Galaxy, and the cosmic background radiation). Our everyday experience is consistent with a stationary Earth.

● **Problem 1–1:** *You are driving at a steady* 100 km h^{-1}. *At noon you pass a parked police car. At twenty minutes past noon, the police car passes you, travelling at* 120 km h^{-1}. *(a) How fast is the police car moving relative to you? (b) When did the police car start driving, assuming that it accelerated from rest to* 120 km h^{-1} *instantaneously? (c) How far away from you was the police car when it started?*

● **Problem 1–2:** *You are walking at* 2 m s^{-1} *down a straight road, which is aligned with the x-axis. At time* $t = 0$ *s you sneeze. At time* $t = 5$ *s a dog barks, and at the moment he barks he is* $x = 10$ *m ahead of you in the road. At time* $t = 10$ *s a car which is just then 15 m behind you ($x = -15$ m) backfires. (a) Plot the*

---

*The sailor is not allowed to use some characteristic rocking or creaking of the boat caused by its motion through the water. That is cheating and anyway it is possible to make a boat which has no such property on a calm sea

[†]Actually, there *are* some observational consequences to the Earth's *rotation* (spin): for example, Foucault's pendulum, the existence of hurricanes and other rotating windstorms, and the preferred direction of rotation of draining water. The point here is that there are no consequences to the Earth's linear motion through space.

positions x and times t of the sneeze, bark and backfire, relative to you, on a two-dimensional graph. Label the points. (b) Plot positions x' and times t' of the sneeze, bark and backfire, relative to an observer standing still, at the position at which you sneezed. Assume your watches are synchronized.

• **Problem 1–3:**   *If you throw a superball at speed v at a wall, it bounces back with the same speed, in the opposite direction. What happens if you throw it at speed v towards a wall which is travelling towards you at speed w? What is your answer in the limit in which w is much larger than v?*

• **Problem 1–4:**   *You are trying to swim directly east across a river flowing south. The river flows at $0.5 \mathrm{~m\,s^{-1}}$ and you can swim, in still water, at $1 \mathrm{~m\,s^{-1}}$. Clearly if you attempt to swim directly east you will drift downstream relative to the bank of the river. (a) What angle θ_a will your velocity vector relative to the bank make with the easterly direction? (b) What will be your speed (magnitude of velocity) v_a relative to the bank? (c) To swim directly east relative to the bank, you need to head upstream. At what angle θ_c do you need to head, again taking east to be the zero of angle? (d) When you swim at this angle, what is your speed v_c relative to the bank?*

1.2   Einstein's principle of relativity

Einstein's principle of relativity says, roughly, that every physical law and fundamental physical constant (including, in particular, the speed of light in vacuum) is the same for all non-accelerating observers. This principle was motivated by electromagnetic theory and in fact the field of special relativity was launched by a paper entitled (in English translation) "On the electrodynamics of moving bodies" (Einstein 1905).[‡] Einstein's principle is not different from Galileo's except that it explicitly states that electromagnetic experiments (such as measurement of the speed of light) will not tell the sailor in the windowless room whether or not the boat is moving, any more than fluid dynamical or gravitational experiments. Since Galileo was thinking of experiments involving bowls of soup and cannonballs dropped from towers, Einstein's principle is effectively a generalization of Galileo's.

The   governing equations of electromagnetism, Maxwell's equations (e.g., Purcell 1985), describe the interactions of magnets, electrical charges and currents, as well as light, which is a disturbance in the electromagnetic field. The equations depend on the speed of light c in vacuum; in other words, if the speed of light in vacuum was different for two different observers, the two observers would be able to tell this simply by performing experiments with magnets, charges and currents. Einstein guessed that a very strong principle of relativity might hold, that is, that the properties of magnets, charges and currents will be the same for all observers, no matter what their relative velocities. Hence the speed of light must be the same for all observers. Einstein's guess was fortified by some experimental evidence available at the time, to be discussed below, and his principle of relativity is now one of the most rigorously tested facts in all of physics, confirmed directly and indirectly in countless experiments.

The consequences of this principle are enormous. In fact, these notes are devoted to the strange predictions and counterintuitive results that follow from it. The most obvious and hardest to accept (though it has been experimentally confirmed countless times now) is that the following simple rule for velocity addition (the rule you must have used to solve the Problems in the previous Section) is false:

Consider a sailor Alejandro (A) sailing past an observer Barbara (B) at speed u. If A throws a cantaloupe, in the same direction as he is sailing past B, at speed v' relative to himself, B will observe the cantaloupe to travel at speed $v = v' + u$ relative to herself. *This rule for velocity addition is wrong.* Or imagine that A drops the cantaloupe into the water and observes the waves travelling forward from the splash. If B is at rest with respect to the water and water waves travel at speed w relative to the water, B will obviously see the waves travel forward from the splash at speed w. On the other hand A, who is moving forward at speed u already, will see the waves travel forward at lower speed $w' = w - u$. *This rule for velocity addition is also wrong!*

After all, instead of throwing a cantaloupe, A could have shined a flashlight. In this case, if we are Galileans (that is, if we believe in the above rule for velocity addition), there are two possible predictions for the speeds at which A and B observe the light to travel from the flashlight. If light is made up of particles which are emitted from their source at speed c relative to the source, then A will observe the light to travel at speed c relative to himself, while B will observe it to travel at $c + u$ relative to herself. If, on the other hand, light is made up of waves that travel at c relative to some medium (analogous to the water for water waves), then we would expect A to see the light travel at $c - u$ and B to see it travel at c (assuming B is at rest with the medium). Things get more complicated if both A and B are moving relative to the medium, but in almost every case we expect A and B to observe different speeds of light if we believe our simple rule for velocity addition.

Einstein's principle requires that A and B observe exactly the same speed of light, so Einstein and the simple rules for velocity addition cannot both be correct. It turns out that Einstein *is* right and the "obvious" rules for velocity addition are incorrect. In this, as in many things we will encounter, our initial intuition is wrong. We will try to build a new, correct intuition based on Einstein's principle of relativity.

• **Problem 1–5:**   *(For discussion.) What assumptions does one naturally make which must be wrong in order for*

[‡]This paper is extremely readable and it is strongly reccomended that the student of relativity read it during a course like this one. It is available in English translation in Lorentz et al. (1923).

A and B to measure the same speed of light in the above example? Consider how speeds are measured: with rulers and clocks.

1.3 The Michelson-Morley experiment

In the late nineteenth century, most physicists were convinced, contra Newton (1730), that light is a wave and not a particle phenomenon. They were convinced by interference experiments whose results can be explained (classically) only in the context of wave optics. The fact that light is a wave implied, to the physicists of the nineteenth century, that there must be a medium in which the waves propagate—there must be something to "wave"—and the speed of light should be measured relative to this medium, called *the aether*. (If all this is not obvious to you, you probably were not brought up in the scientific atmosphere of the nineteenth century!) The Earth orbits the Sun, so it cannot be at rest with respect to the medium, at least not on every day of the year, and probably not on any day. The motion of the Earth through the aether can be measured with a simple experiment that compares the speed of light in perpendicular directions. This is known as the Michelson-Morley experiment and its surprising result was a crucial hint for Einstein and his contemporaries in developing special relativity.[§]

Imagine that the hypothesis of the aether is correct, that is, there is a medium in the rest frame of which light travels at speed c, and Einstein's principle of relativity does not hold. Imagine further that we are performing an experiment to measure the speed of light $c_\oplus$ on the Earth, which is moving at velocity $\boldsymbol{v}_\oplus$ (a vector with magnitude $v_\oplus$) with respect to this medium. If we measure the speed of light in the direction parallel to the Earth's velocity $\boldsymbol{v}_\oplus$, we get $c_\oplus = c - v_\oplus$ because the Earth is "chasing" the light. If we measure the speed of light in the opposite direction—antiparallel to the Earth's velocity—we get $c_\oplus = c + v_\oplus$. If we measure in the direction perpendicular to the motion, we get $c_\oplus = \sqrt{c^2 - v_\oplus^2}$ because the speed of light is the hypotenuse of a right triangle with sides of length $c_\oplus$ and $v_\oplus$.[¶] If the aether hypothesis is correct, these arguments show that the motion of the Earth through the aether can be detected with a laboratory experiment.

The Michelson-Morley experiment was designed to perform this determination, by comparing directly the speed of light in perpendicular directions. Because it is very difficult to make a direct measurement of the speed of light, the device was very cleverly designed to make an accurate *relative* determination. Light entering the apparatus from a lamp is split into two at a half-silvered mirror. One half of the light bounces back and forth 14 times in one direction and the other half bounces back and forth 14 times in the perpendicular direction; the total distance travelled is about 11 m per beam. The

two beams are recombined and the interference pattern is observed through a telescope at the output. The whole apparatus is mounted on a stone platform which is floated on mercury to stabilize it and allow it to be easily rotated. Figure 1.1 shows the apparatus, and Figure 1.2 shows a simplified version.
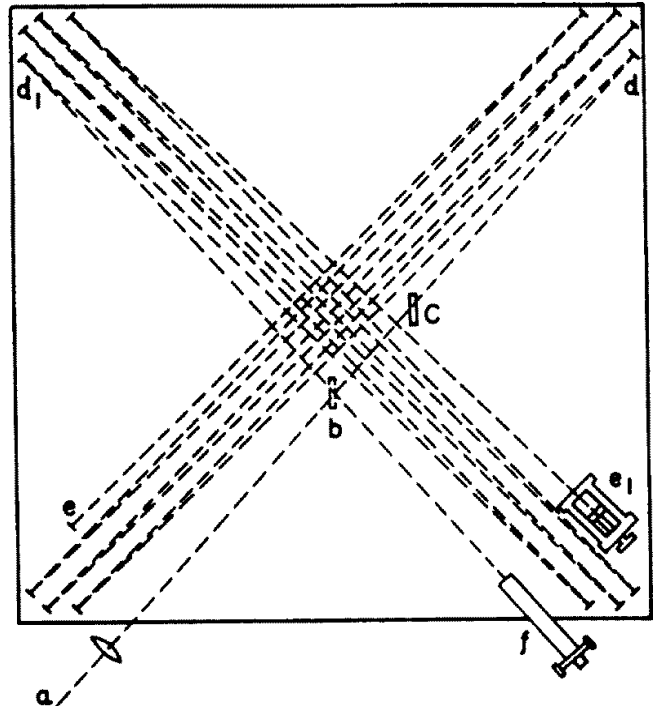

Figure 1.1: The Michelson-Morley apparatus (from Michelson & Morley 1887). The light enters the apparatus at a, is split by the beam splitter at b, bounces back and forth between mirrors d and e, d_1 and e_1, with mirror e_1 adjustable to make both paths of equal length, the light is recombined again at b and observed through the telescope at f. A plate of glass c compensates, in the direct beam, for the extra light travel time of the reflected beam travelling through the beam splitter an extra pair of times. See Figure 1.2 for a simplified version.

If the total length of travel of each beam is ℓ and one beam is aligned parallel to $\boldsymbol{v}_\oplus$ and the other is aligned perpendicular, the travel time in the parallel beam will be

$$t_\parallel = \frac{\ell}{2\left(c + v_\oplus\right)} + \frac{\ell}{2\left(c - v_\oplus\right)} = \frac{\ell\,c}{c^2 - v_\oplus^2} \qquad (1.1)$$

because half the journey is made "upstream" and half "downstream." In the perpendicular beam,

$$t_\perp = \frac{\ell}{\sqrt{c^2 - v_\oplus^2}} \qquad (1.2)$$

because the whole journey is made at the perpendicular velocity. Defining $\beta \equiv v_\oplus/c$ and pulling out common factors, the difference in travel time, between parallel and

---

[§]The information in this section comes from Michelson & Morley (1887) and the history of the experiment by Shankland (1964).

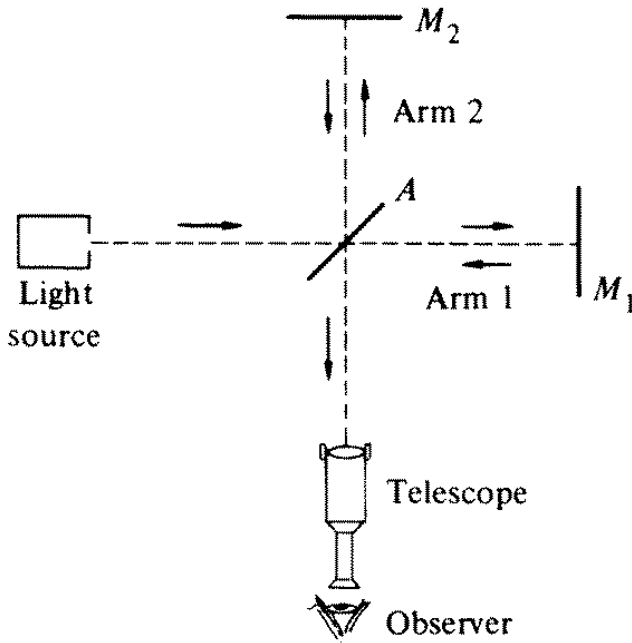[¶]The demonstration of this is left as an exercise.

Figure 1.2: The Michelson apparatus (from Kleppner & Kolenkow 1973), the predecessor to the Michelson-Morley apparatus (Figure 1.1). The Michelson apparatus shows more clearly the essential principle, although it is less sensitive than the Michelson-Morley apparatus because the path length is shorter.

perpendicular beams, is

$$\Delta t = \frac{\ell}{c} \left[\frac{1}{1 - \beta^2} - \frac{1}{\sqrt{1 - \beta^2}} \right] \qquad (1.3)$$

For small x, $(1+x)^n \approx 1 + n\,x$, so

$$\Delta t \approx \frac{\ell}{2c}\,\beta^2 \qquad (1.4)$$

Since the apparatus will be rotated, the device will swing from having one arm parallel to the motion of the Earth and the other perpendicular to having the one perpendicular and the other parallel. So as the device is rotated through a half turn, the time delay between arms will change by twice the above Δt.

The lateral position of the interference fringes as measured in the telescope is a function of the relative travel times of the light beams along the two paths. When the travel times are equal, the central fringe lies exactly in the center of the telescope field. If the paths differ by one-half a period (one-half a wavelength in distance units), the fringes shift by one-half of the fringe separation, which is well resolved in the telescope. As the apparatus is rotated with respect to the Earth's motion through the aether, the relative travel times of the light along the two paths was expected to change by 0.4 periods, because (in the aether model) the speed of light depends on direction. The expected shift of the interference fringes was 0.4 fringe spacings, but no shift at all was observed as the experimenters rotated the apparatus. Michelson and

Morley were therefore able to place upper limits on the speed of the Earth $v_\oplus$ through the aether; the upper limits were much lower than the expected speed simply due to the Earth's orbit around the Sun (let alone the Sun's orbit around the Galaxy and the Galaxy's motion among its neighboring galaxies).

Michelson and Morley concluded that something was wrong with the standard aether theory; for instance, perhaps the Earth drags its local aether along with it, so we are always immersed in locally stationary aether. In a famous paper, Lorentz (1904) proposed that all moving bodies are contracted along the direction of their motion by the amount exactly necessary for the Michelson-Morley result to be null. Both these ideas seemed too much like "fine-tuning" a so-far unsubstatiated theory.

Einstein's explanation—that there is no aether and that the speed of light is the same for all observers (and in all directions)—is the explanation that won out eventually.[||] The Michelson-Morley experiment was an attempt by "sailors" (Michelson and Morley) to determine the speed of their "boat" (the Earth) without looking out the window or comparing to some other object, so according to the principle of relativity, they were doomed to failure.

● **Problem 1–6:** *With perfect mirrors and light source, the Michelson-Morley apparatus can be made more sensitive by making the path lengths longer. Why is a device with longer paths more sensitive? The paths can be lengthened by making the platform larger or adding more mirrors (see Figure 1.1). In what ways would such modification also degrade the performance of the device given imperfect mirrors and light source (and other real-world concerns)? Discuss the pros and cons of such modifications.*

● **Problem 1–7:** *Show that under the hypothesis of a stationary aether, the speed of light as observed from a platform moving at speed v, in the direction perpendicular to the platform's motion, is $\sqrt{c^2 - v^2}$. For a greater challenge: what is the observed speed for an arbitrary angle θ between the direction of motion and the direction in which the speed of light is measured? Your answer should reduce to $c + v$ and $c - v$ for $\theta = 0$ and π.*

It is worthy of note that when Michelson and Morley first designed their experiment and predicted the fringe shift, they did not realize that the speed of light perpendicular to the direction of motion of the platform would be other than c. This correction was pointed out to them by Potier in 1881 (Michelson & Morley, 1887).

[||]It was also Poincaré's (1900) explanation. Forshadowing Einstein, he said that the Michelson-Morley experiment shows that absolute motion cannot be detected to second order in v/c and so perhaps it cannot be detected to any order. Poincaré is also allegedly the first person to have named this proposal a "principle of relativity."

1.4   The "specialness" of special relativity

Why is this subject called "special relativity," and not just "relativity"? It is because the predictions we make only strictly hold in certain special situations.

Some of the thought experiments (and real experiments) described in these notes take place on the surface of the Earth, and we will assume that the gravitational field of the Earth (and all other planets and stars) is negligible.** The laws of special relativity strictly hold only in a "freely falling" reference frame in which the observers experience no gravity. The laws strictly hold when we are falling towards the Earth (as in a broken elevator; e.g., Frautschi et al., 1986, ch. 9) or orbiting around the Earth (as in the Space Shuttle; ibid.), but not when we are standing on it.

Does the gravitational field of the Sun affect our results? No, because we are orbiting the Sun. The Earth is in a type of "free fall" around the Sun. Does the rotation of the Earth affect our results? Yes, because the centrifugal force that is felt at the equator is equivalent to an outward gravitational force. However, this effect is much smaller than the Earth's gravity, so it is even more negligible.

In addition, we are going to assume that all light signals are travelling in vacuum. The speed of light in air is actually a bit less than the speed of light in vacuum. We will neglect this difference. The "c" that comes into the general equations that we will derive is the speed of light in vacuum, no matter what the speed at which light is actually travelling in the local medium. Everything is simpler if we just treat all our experiments as if they are occurring in vacuum.

• **Problem 1–8:**   (Library excercise.) *How much slower (or faster) is the speed of light in air, relative to vacuum? How do you think the speed will depend on temperature and pressure? How much slower (or faster) is the speed of light in glass and water, relative to vacuum. Give your references.*

• **Problem 1–9:**   *You shine a flashlight from one end zone of a football field to a friend standing in the other end zone. Because of the Earth's gravity, the beam of light will be pulled downwards as it travels across the field. Estimate, any way you can, the distance the light will "drop" vertically as it travels across the field. What deflection angle does this correspond to, in arcseconds?*

Don't worry about getting a precise answer, just estimate the order of magnitude of the effect.

**The fractional error that the Earth's gravity introduces into the experiments we describe must depend only on the acceleration due to gravity g, the parameters of each experiment, and fundamental constants. Fractional error is dimensionless, and the most obvious fundamental constant to use is c. The ratio g/c has dimensions of inverse time. This suggests that an experiment which has a characteristic time τ or length ℓ will not agree with the predictions of special relativity to better than a fractional error of about $\tau\, g/c$ or $\ell\, g/c^2$ if it is performed on the surface of the Earth.

Chapter 2

Time dilation and length contraction

This Chapter is intended to demonstrate the simplicity of special relativity. With one basic thought experiment the two most important effects predicted by the theory are derived: time dilation and length contraction. For the beginning student of relativity, this is the most important chapter.

It is emphasized that the predicted effects are real, not just apparent.

Before starting, recall Einstein's (1905) principle of relativity (hereafter "the" principle of relativity): there is no preferred reference frame; no entirely on-board experiment can tell a sailor the speed of her or his boat. Its first consequence is that the speed of light is the same in all frames.

2.1 Time dilation

Consider two observers, Deepto (D) and Erika (E), moving relative to one another in spaceships. D measures E's speed to be u with respect to D's rest frame. By symmetry, E must also measure D's speed to be u with respect to E's rest frame. If this is not obvious to you, notice that there is no absolute difference between D and E. If they did not measure the same speed, which one of them would measure a higher speed? In order for one to measure a higher speed, one of them would have to be in a special or "preferred" frame; the principle of relativity precludes this.

Now imagine that D and E each carry a clock of a certain very strange type. These "light-clocks" consist of an evacuated glass tube containing a lightbulb, a mirror, a photodetector and some electrical equipment. The photodetector is right next to the lightbulb (but separated by a light-blocking shield) and the mirror is 0.5 m from the lightbulb (see Figure 2.1). When the clock is started, the lightbulb flashes, light bounces off the mirror and back into the photodetector. When the photodetector registers the light, it immediately signals the lightbulb to flash again. Every time the photodetector registers a light pulse, it flashes the bulb again.

The round-trip distance for the light inside the light-clock is 1 m and the speed of light c is roughly 3×10^8 m s^{-1}, so the round-trip time for the light is roughly 3.3×10^{-9} s. The clock ticks off time in 3.3 ns (nanosecond) intervals. The speed of light is the same for all
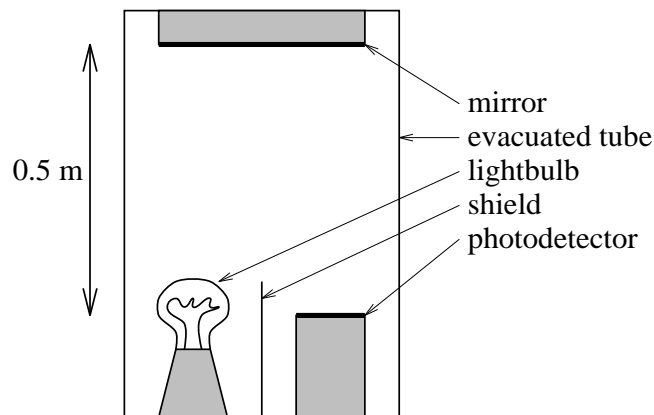

Figure 2.1: The schematic layout of a light-clock. The round-trip distance (lightbulb to mirror to photodetector) for the light is 1 m.

observers, so c can be seen as a conversion factor between time and distance. Under this interpretation, the clock ticks off time in meters![*]

Imagine that D holds his light-clock so that the light is bouncing back and forth at right angles to his direction of motion with respect to E. D observes the light flashes in his clock to make 1 m round trips in $\Delta t = 3.3$ ns intervals. What does E observe? Recall that D is moving at speed u with respect to E, so in E's rest frame the light in D's clock is not really making round trips. As it travels down the tube and back, D is advancing in the perpendicular direction; the light takes a zig-zag path which is longer than the straight back-and-forth path (see Figure 2.2). By the principle of relativity, E and D must observe the same speed of light, so we are forced to conclude that E will measure[†] longer time intervals $\Delta t'$ between the flashes in D's clock than D will. (In this chapter, all quantities that E measures will be primed and all that D measures will be unprimed.) What is the difference between Δt and $\Delta t'$?

In E's rest frame, in time $\Delta t'$, D advances a distance

---

[*]Now that the meter is defined in terms of the second, this is in fact the interpretation of the speed of light that the International Standards Organization accepts. The speed of light is *defined* to be 2.99792458×10^8 m s^{-1}.

[†]What is meant by "measure" here is explained in the next Section—Erika is a very good scientist!
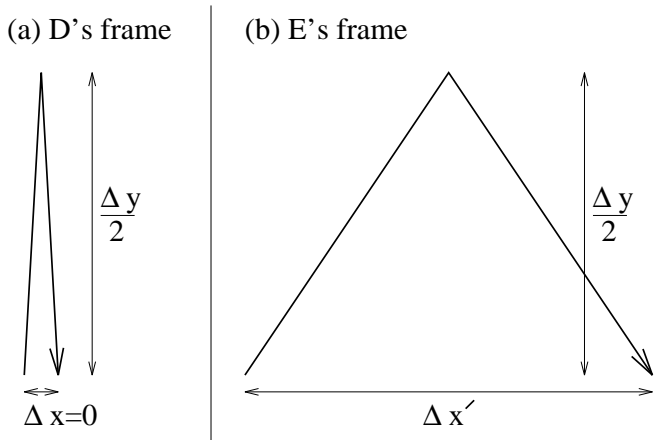
Figure 2.2: The trajectory of the light in D's light-clock, as observed by (a) D and (b) E. Note that the light follows a longer path in E's frame, so E measures a longer time interval $\Delta t'$.

$\Delta x' = u \, \Delta t'$ and the light in D's clock must go a total distance $\Delta \ell' = c \, \Delta t'$. By the Pythagorean theorem $(\Delta \ell')^2 = (\Delta x')^2 + (\Delta y)^2$, where Δy is the total round-trip length of the clock (1 m in this case) in its rest frame and for now it has been assumed that $\Delta y = \Delta y'$ (this will be shown in Section 2.3). Since $\Delta y = \Delta \ell = c \, \Delta t$, we find

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{u^2}{c^2}}} \qquad (2.1)$$

The time intervals between flashes of D's clock are longer as measured by E than as measured by D. This effect is called *time dilation.* Moving clocks go slow.

It is customary to define the dimensionless speed β and the *Lorentz factor* γ by

$$\beta \equiv \frac{u}{c} \qquad (2.2)$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} \qquad (2.3)$$

Because (as we shall see later) nothing travels faster than the speed of light, u is always less than c, so $0 \leq \beta < 1$, and $\gamma \geq 1$. Using these new symbols, $\Delta t' = \gamma \, \Delta t$.

Above we found that "moving clocks go slow," but one might object that we have shown only that these strange light-clocks go slow. However, we can show that *all* clocks are subject to the same time dilation. Suppose that in addition to his light-clock, D also has a wristwatch that ticks every 3.3 ns, and suppose (incorrectly) that this watch is not subject to time dilation; i.e., suppose that E observes the watch to tick with intervals of 3.3 ns no matter what D's speed. When D is not moving with respect to E the wristwatch and light-clock tick at the same rate, but when D is moving at high speed, they tick at different rates because, by supposition, one is time-dilated and the other is not. D could use the relative tick rates of the watch and clock to determine his speed, and thereby violate the principle of relativity. It is left to the ambitious reader to

prove that it is not possible for D to observe both timepieces to tick at the same rate while E observes them to tick at different rates.

The reader might object that we have already violated relativity: if D and E are in symmetric situations, how come E measures longer time intervals? We must be careful. E measures longer time intervals for D's clock than D does. By relativity, it must be that D *also* measures longer time intervals for E's clock than E does. Indeed this is true; after all, all of the above arguments are equally applicable if we swap D and E. This is the fundamentally counterintuitive aspect of relativity. How it can be that both observers measure slower rates on the other's clock? The fact is, there is no contradiction, as long as we are willing to give up on a concept of absolute time, agreed-upon by all observers. The next two Chapters explore this and attempt to help develop a new intuition.

• **Problem 2–1:** *Your wristwatch ticks once per second. What is the time interval between ticks when your wristwatch is hurled past you at half the speed of light?*

• **Problem 2–2:** *How fast does a clock have to move to be ticking at one tenth of its rest tick rate? One one-hundredth? One one-thousandth? Express your answers in terms of the difference $1 - \beta$, where of course $\beta \equiv v/c$.*

• **Problem 2–3:** *Consider the limit in which $\gamma \gg 1$, so its inverse $1/\gamma$ is a small number. Derive an approximation for β of the form $\beta \approx 1 - \epsilon$ which is correct to second order in $1/\gamma$.*

• **Problem 2–4:** *Consider the low-speed limit, in which $\beta \ll 1$. Derive an expression for γ of the form $\gamma \approx 1 + \epsilon$ which is correct to second order in β.*

• **Problem 2–5:** *Prove (by thought experiment) that it is not possible for D to observe both his light-clock and his wristwatch to tick at the same rate while E observes them to tick at different rates. (Hint: Imagine that both of D's clocks punch a ticker tape and the experimentalists compare the tapes after the experiment is over.)*

2.2   Observing time dilation

In the previous section, as in the rest of these notes, it is important to distinguish between what an ideally knowledgeable observer *observes* and what an ordinary person *sees.* As much as possible, the term "to observe" will be used to mean "to measure a real effect with a correct experimental technique," while "to see" will be reserved for apparent effects, or phenomena which relate to the fact that we look from a particular viewpoint with a particular pair of eyes. This means that we won't talk about what is "seen" in detail until Chapter 7.

Though E *observes* D's clock to run slow, what she *sees* can be quite different. The time intervals between the flashes of D's clock that she sees depends on the time dilation *and* the changing path lengths that the light tra-

verses in getting to E. The path lengths are changing because D is moving with respect to E (see Figure 2.3). In order to correctly measure the rate of D's clock, E must subtract the light-travel time of each pulse (which she can compute by comparing the direction from which the light comes with the trajectory that was agreed upon in advance). It is only when she subtracts these time delays that she measures the time between ticks correctly, and when she does this, she will find that the time between ticks is indeed $\Delta t'$, the dilated time.
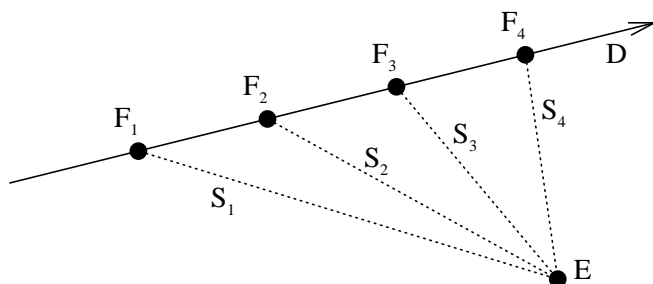

Figure 2.3: Observing the time delay. Because D is moving with respect to E, the flashes (F_1 through F_4) from his clock travel along paths (S_1 through S_4) of different lengths in getting to E. Hence different flashes take different times to get to E. E must correct for this before making any statements about time dilation. It is after the correction is made that E observes the predicted time dilation.

● **Problem 2–6:** *Consider a clock, which when at rest produces a flash of light every second, moving away from you at $(4/5)c$. (a) How frequently does it flash when it is moving at $(4/5)c$? (b) By how much does distance between you and the clock increase between flashes? (c) How much longer does it take each flash to get to your eye than the previous one? (d) What, therefore, is the interval between the flashes you see?*

You will find that the time interval between the flashes you see is much longer than merely what time-dilation predicts, because successive flashes come from further and further away. This effect is known as the Doppler shift and is covered in much more detail in Chapter 7

2.3 Length contraction

Imagine that E observes D's clock to tick 100 times during a journey from planet A to planet B, two planets at rest in E's rest frame.

D must also observe 100 ticks during this same journey. After all, if we imagine that the clock punches a time card each time it ticks and D inserts the time card at point A and removes it at point B, it must have been punched a definite number of times when it is removed. D and E must agree on this number, because, for example, they can meet later and examine the card.

In addition to agreeing on the number of ticks, D and E also agree on their relative speed. (They must, because there is total symmetry between them: if one measured

a larger speed, which one could it be?) However, they do *not* agree on the rate at which D's clock ticks. While E measures the distance between A and B to be $\ell' = 100\,u\,\Delta t'$, D measures it to be $\ell = 100\,u\,\Delta t = \ell'/\gamma$. Since $\gamma > 1$, D measures a *shorter* distance than E. D is moving relative to the planets A and B, while E is stationary. Planets A and B can be thought of as being at the ends of a ruler stick which E is holding, a ruler stick which is moving with respect to D. We conclude that moving ruler sticks are shortened; this effect is *length contraction,* or sometimes *Lorentz contraction.*

It is simple to show that length contraction acts only parallel to the direction of motion. Imagine that both E and D are carrying identical pipes, aligned with the direction of their relative motion (see Figure 2.4). Let
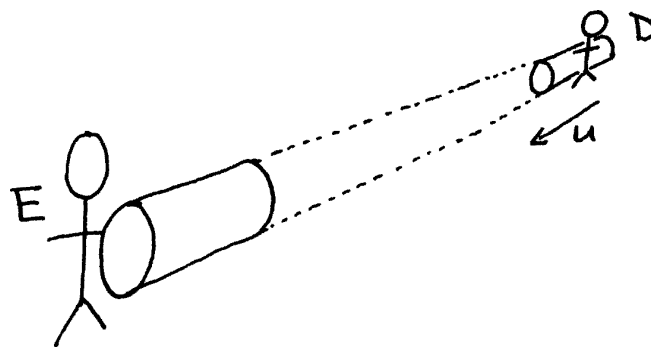

Figure 2.4: E and D carrying pipes to prove that there can be no length changes perpendicular to the direction of motion.

us assume (incorrectly) that the large relative velocity causes the diameter of E's pipe to contract in D's frame. If this happens, D's pipe becomes larger than E's pipe, so E's pipe "fits inside" D's pipe. But E and D are interchangeable, so D's pipe contracts in E's frame and D's pipe fits inside E's. Clearly it cannot be that both D's fits inside E's and E's fits inside D's, so there is a contradiction; there can be no length changes perpendicular to the direction of relative motion.

Note that because there are no length changes perpendicular to the direction of motion, we cannot explain away time dilation and length contraction with length changes in the light-clock perpendicular to the direction of motion.

● **Problem 2–7:** *How fast do you have to throw a meter stick to make it one-third its rest length?*

● **Problem 2–8:** *Two spaceships, each measuring 100 m in its own rest frame, pass by each other traveling in opposite directions. Instruments on board spaceship A determine that the front of spaceship B requires 5×10^{-6} s to traverse the full length of A. (a) What is the relative velocity v of the two spaceships? (b) How much time elapses on a clock on spaceship B as it traverses the full length of A? (From French 1966.)*

● **Problem 2–9:** *That there can be no length contraction perpendicular to the direction of motion is often demonstrated with the example of a train and its track;*

i.e., if there were length changes perpendicular the train would no longer fit on the track. Make this argument, and in particular, explain why the train must fit on the track no matter how fast it is going.

2.4   Magnitude of the effects

As these example problems show, the effects of time dilation and length contraction are extremely small in everyday life, but large for high-energy particles and any practical means of interstellar travel.

• **Problem 2–10:**   *In the rest frame of the Earth, the distance ℓ between New York and Los Angeles is roughly 4000 km. By how much is the distance shortened when observed from a jetliner flying between the cities? From the Space Shuttle? From a cosmic ray proton traveling at 0.9c?*

In the rest frame, the distance is ℓ; to an observer traveling at speed u along the line joining the cities, it is $\ell' = \ell/\gamma$. The difference is

$$\ell - \ell' = \left(1 - \frac{1}{\gamma}\right)\ell = \left(1 - \sqrt{1 - \beta^2}\right)\ell \qquad (2.4)$$

For ϵ much smaller than unity, $(1 + \epsilon)^n \approx 1 + n\,\epsilon$, so for speeds $u \ll c$ or $\beta \ll 1$, we have

$$\ell - \ell' \approx \frac{1}{2}\,\beta^2\,\ell \qquad (2.5)$$

A jetliner takes about 6 h to travel from New York to Los Angeles, so its speed is roughly $u = 4000/6$ km h^{-1} or $\beta = 6 \times 10^{-7}$. Since $\beta \ll 1$, we have that $\ell - \ell' \approx 8 \times 10^{-7}$ m, or 0.8 microns! The Space Shuttle takes about 1.5 h to orbit the earth, on an orbit with radius roughly 6500 km, so $\beta = 2.5 \times 10^{-5}$. Here $\ell - \ell' \approx 1.3$ mm.

As for the cosmic ray proton, $\beta = 0.9$, so it is no longer true that $\beta \ll 1$; we gain nothing by using the approximation. We find $\gamma = 2.3$ and so $\ell - \ell' = 2300$ km.

• **Problem 2–11:**   *At rest in the laboratory, muons have a mean life T of 2.2×10^{-6} s or 2.2 µs, or in other words, the average time a muon exists from production (in a collision, say) to decay (into an electron and neutrinos) is 2.2 µs (Particle Data Group, 1994). If, as experimentalists, we need a sample of muons to have a longer mean life of $T' = 11$ µs, to what speed u must we accelerate them? What distance ℓ, on average, does one of these high-speed muons travel before decaying?*

We want the muons to age at 1/5 their usual rate, so we want time dilation by a factor $\gamma = 5$. Inverting the formula for γ we find

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \qquad (2.6)$$

or in this case $\beta = 24/25$. This makes $u = 24c/25$ and $\ell = u\,T = 630$ m.

• **Problem 2–12:**   *Alpha Centauri is a distance of $\ell = 4.34$ light years (one light year is the distance light travels in one year) from the Earth. At what speed u must a 25-year-old astronaut travel there and back if he or she is to return before reaching age 45? By how much will the astronaut's siblings age over the same time?*

This is the famous "twin paradox," which we will cover in gory detail in Section 4.5. For now, let us be simplistic and answer the questions without thinking.

We want the elapsed time T' in the astronaut's frame to be 20 years as he or she goes a distance $2\ell'$, the distance from the Earth to Alpha Centauri and back in the astronaut's frame. The time and distance are related by $T' = 2\ell'/u = 2\ell/(\gamma u)$. So we need $\gamma u = 2\ell/T'$. Dividing by c, squaring and expanding we need

$$\frac{\beta^2}{1 - \beta^2} = \left(\frac{2\,\ell}{c\,T'}\right)^2 = (0.434)^2 \qquad (2.7)$$

This is a linear equation for β^2; we find $\beta = 0.398$. So the astronaut must travel at $u = 0.398c$, and from the point of view of the siblings, the trip takes $T = 2\ell/u = 21.8$ yr.

2.5   Experimental confirmation

As we have seen in the previous section, the effects of time dilation and length contraction are not very big in our everyday experience. However, these predictions of special relativity *have* been confirmed experimentally. Time dilation is generally easier to confirm directly because Nature provides us with an abundance of moving clocks, and because in such experiments, it is generally more straightforward to design procedures in which the delays from light travel time (discussed in Section 2.2) are not important. Of course in addition to experiments like the one discussed in this section, both time dilation and length contraction are confirmed indirectly countless times every day in high energy physics experiments around the world.

The first direct confirmation of time dilation was obtained by Bruno Rossi and David Hall, studying the decay of muons (in those days called "mesotrons" or "mu mesons") as they descend through the Earth's atmosphere.[‡] Muons are elementary particles[§] produced at high altitude when cosmic rays (fast-moving protons and other atomic nuclei) collide with atoms in the Earth's atmosphere. When produced more or less at rest in the laboratory, each muon has a mean lifetime of $\tau_0 = 2.5 \times 10^{-6}$ seconds before it disintegrates. Indeed, if one has N_0 muons at time zero and then looks at a later time t, the number of muons will have dropped to $N(t) = N_0\,e^{-t/\tau_0}$. If there were no such thing as time dilation, the mean distance a muon moving at high speed $v = \beta c$ could travel before disintegrating would be $L = v\tau_0$. Similarly if at position zero one has N_0 muons moving at speed v down a tube, at a position x further

---

[‡]The information in this section comes from Rossi & Hall (1941), their extremely readable, original paper.

[§]For those who care, muons are *leptons,* most analogous to electrons, with the same charge but considerably more mass. They are unstable and typically decay into electrons and neutrinos.

down the tube there would be only $N(x) = N_0 \, e^{-x/L}$. As the speed of the muons approaches c, the mean range would approach $c\tau_0$, or 750 m. Since the muons are created at high altitude, very few of them could reach the ground.

However, we expect that time dilation *does* occur, and so the mean life τ and range L of the moving muons will be increased by the Lorentz factor $\gamma \equiv (1 - \beta^2)^{-1/2}$ to $\tau = \gamma \, \tau_0$ and $L = \gamma \, v \, \tau_0$. Although all the muons will be moving at speeds close to c (β nearly 1), they will have different particular values of γ and therefore decay with different mean ranges. Bruno & Rossi measure the fluxes (number of muons falling on a detector of a certain area per minute) of muons of two different kinetic energies at observing stations in Denver and Echo Lake, Colorado, separated in altitude by $\Delta h = 1624$ m (Denver below Echo Lake). The higher-energy muons in their experiment have Lorentz factor $\gamma_1 \approx 18.8$ (speed $v_1 \approx 0.9986c$) while the lower-energy muons have $\gamma_2 \approx 6.3$ ($v_2 \approx 0.987c$). Because we expect the mean range L of a muon to be $L = \gamma \, v \, \tau_0$, we expect the ratio of ranges L_1/L_2 for the two populations of muons to be $(\gamma_1 \, v_1)/(\gamma_2 \, v_2) \approx 3.0$. The flux of higher-energy muons at Denver is lower by a factor of 0.883 ± 0.005 than it is at Echo Lake, meaning that if they have mean range L_1, $e^{-\Delta h/L_1} = 0.883$. The flux of lower-energy muons decreases by a factor of 0.698 ± 0.002, so $e^{-\Delta h/L_2} = 0.698$. Taking logarithms and ratios, we find that $L_1/L_2 = 2.89$ as predicted. The results do not make sense if the time dilation factor (the Lorentz factor) is ignored.

• **Problem 2–13:** *Consider a muon traveling straight down towards the surface of the Earth at Lorentz factor $\gamma_1 \approx 18.8$. (a) What is the vertical distance between Denver and Echo Lake, according to the muon? (b) How long does it take the muon to traverse this distance, according to the muon? (c) What is the muon's mean lifetime, according to the muon? (d) Answer the above parts again but now for a muon traveling at Lorentz factor $\gamma_2 \approx 6.3$.*

• **Problem 2–14:** *Charged pions are produced in high-energy collisions between protons and neutrons. They decay in their own rest frame according to the law*

$$N(t) = N_0 \, 2^{-t/T} \tag{2.8}$$

where $T = 2 \times 10^{-8}$ s is the half-life. A burst of pions is produced at the target of an accelerator and it is observed that two-thirds of them survive at a distance of 30 m from the target. At what γ value are the pions moving? (From French 1966.)

• **Problem 2–15:** *A beam of unstable K^+ mesons, traveling at speed $\beta = \sqrt{3}/2$, passes through two counters 9 m apart. The particles suffer a negligible loss of speed and energy in passing through the counters but give electrical pulses that can be counted. It is observed that 1000 counts are recorded in the first counter and 250 in the second. Assuming that this whole decrease is due to decay of the particles in flight, what is their half-life as it would be measured in their own rest frame? (From French 1966.)*

Chapter 3

The geometry of spacetime

Observers in different frames of reference, even if they are observing identical events, may observe very different relationships *between* those events. For example, two events which are simultaneous for one observer will not, in general, be simultaneous for another observer. However, the principle of relativity must hold, i.e., both observers must agree on all laws of physics and in particular on the speed of light. This principle allows detailed construction of the differences between two observers' measurements as a function of their relative velocity. In this chapter we derive some of these relationships using a very useful tool: the spacetime diagram. With spacetime diagrams most special relativity problems are reduced to simple geometry problems. The geometric approach is the most elegant method of solving special relativity problems and it is also the most robust because it requires the problem-solver to visualize the relationships between events and worldlines.

3.1 Spacetime diagrams

Frances (F) and Gregory (G) live on planets A and B, respectively, separated in space by $\ell = 6 \times 10^{11}$ m (600 million km). Exactly halfway between their home planets, on the line joining them, is an interplanetary café (C), at which they decide to meet at noon. F has a standard-model spaceship which travels at speed $c/5$ (which corresponds to $\beta = 1/5$), while G's sporty model travels at $c/3$ ($\beta = 1/3$). If we choose a coordinate system with the x-axis pointing along the direction from A to B, we can plot the trajectories, or *worldlines,* of F and G on a diagram with distance x on the abscissa and time t on the ordinate. Actually, to emphasize the geometry of special relativity, we will use not t to mark time, but ct, which has dimensions of distance.* Such a plot, as in Figure 3.1, is a *spacetime diagram.* Figure 3.1 is clearly drawn in the rest frame of planets A and B: the planet worldlines are vertical; the planets do not change position with time.

They meet at noon at the café. Their meeting is an *event:* it takes place in a certain location, at a certain time. Anything that has both a position and a time is an event. For example, the signing of the United States' Declaration of Independence was an event: it took place on 4 July, 1776, and it took place in Philadelphia, Penn-

sylvania. Each tick of a clock is an event: it happens at a given time at the location of the clock. Events are 3+1-dimensional[†] points—they have three spatial coordinates and one time coordinate. In the case of the meeting M at the café of F and G, we needed only 1+1 dimensions to specify it because we began by restricting all activity to the x-axis, but in general 3+1 dimensions are needed. On Figure 3.1, event M is marked, along with two other events K and L, the departures of F and G.

Because we are marking time in dimensions of distance ct, the inverse slope $\Delta x/(c\Delta t)$ of a worldline at some time ct is the speed of the corresponding object in units of c, or in other words, β. As we will see below, nothing can travel faster than the speed of light. So, all worldlines must be steeper than $45°$ on the spacetime diagram, except, of course, for the worldlines of flashes of light or photons, which have exactly $45°$ worldlines. Radio, infrared, optical, ultraviolet, x-ray and gamma-ray signals all travel on $45°$ worldlines maybe neutrinos do too[‡].

• **Problem 3–1:** *The next day F decides to meet G at the café again, but realizes that she did not arrange this with G in advance. She decides to send a radio message that will get to G at exactly the time he should depart. When should F send this message?*

We can answer this problem trivially by looking at the spacetime diagram. If we drop a $45°$ line from event L, G's departure, going back in time towards planet A, we can find the event at which it intersects F's worldline. This is done in Figure 3.2; we see that it intersects F's worldline exactly at event K, the time of her departure. This means that F should send the radio message at exactly the time she departs for the café.

3.2 Boosting: changing reference frames

Heather (H) and Juan (J) are two more residents of planets A and B respectively. (A and B are separated by $\ell = 6 \times 10^{11}$ m in the x-direction.) Early in the morning (at event P) H sends J a radio message. At event Q, J receives the message. A time τ later in the day, H sends J

---

*Recall the idea, from Section 2.1, that c is merely a conversion factor between time and distance.

[†]One could say "4-dimensional," but it is customary among relativists to separate the numbers of space and time dimensions by a plus sign. The reason for this will be touched upon later.

[‡]As we will see in Chapter 6, neutrinos travel at the speed of light only if they are massless; this is currently a subject of debate.
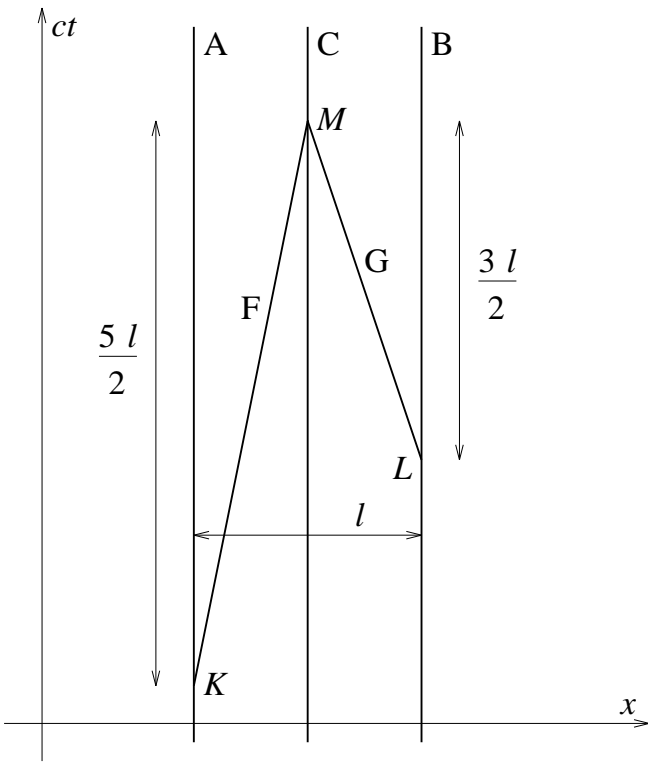
Figure 3.1: Worldlines of F and G meeting at the café, and worldlines of their home planets A and B, and the café itself, C. The event of F's departure is K, of G's is L, and of their meeting is M. This diagram is in the rest frame of A, B, and C because these objects have vertical worldlines. Note that the time (vertical) axis is marked in units of distance ct.

Figure 3.2: When should F send the radio message to G? By dropping a $45°$ line (dotted) from event L to F's worldline, we find that she should send it right when she departs; at event K.

another message at event R, and J receives it at event S. The spacetime diagram with these events and the worldlines of H, J and the messages is shown in Figure 3.3. The diagram is drawn in what we will call "H's frame" or "H's rest frame," because it is a reference frame in which H is at rest.

While this is all going on, Keiko (K) is travelling at speed u between planets A and B. How do we re-draw the spacetime diagram in K's frame, a reference frame in which K is at rest? First of all, K is moving at speed u relative to H and J, so in K's frame H and J will be moving at speed $-u$. Thus, H's and J's worldlines in K's frame will have equal but opposite slope to that of K's worldline in H's frame. Time dilation (Section 2.1) says that moving clocks go slow, so in K's frame, events P and R will be separated in time not by τ but by $\Delta t' = \gamma \tau$. Same for Q and S. (All quantities in K's frame will be primed.) Length contraction (Section 2.3) says that moving ruler sticks are shortened. This means that the distance separating the parallel worldlines of two objects moving at the same speed (the "ends of the ruler stick") is shorter by a factor $1/\gamma$ in a frame moving at speed u than it is in the frame at which the two objects are at rest. H and J, therefore, are separated by not ℓ but $\Delta x' = \ell/\gamma$ in the horizontal direction. Einstein's principle of relativity
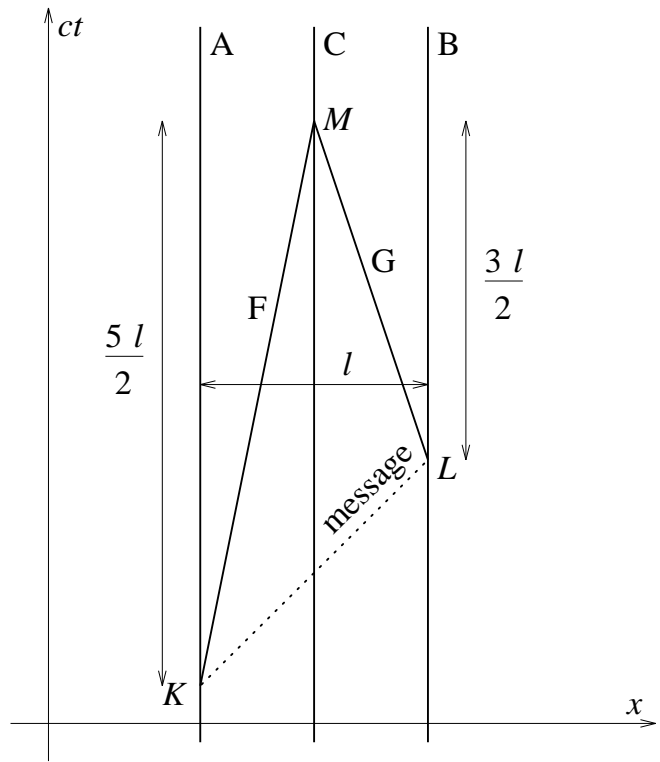
says that the speed of light is the same in both frames, so *the radio signals will still have $45°$ worldlines.* Thus, the spacetime diagram in K's frame must be that pictured in Figure 3.4.

The transformation from H's frame to K's is a *boost transformation* because it involves changing velocity. The boost transformation is central to special relativity; it is the subject of this and the next chapter.

● **Problem 3–2:** *Re-draw the events and worldlines of Figures 3.3 and 3.4 from the point of view of an observer moving at the same speed as K relative to H and J but in the opposite direction.*

● **Problem 3–3:** *A rocket ship of proper length ℓ_0 travels at constant speed v in the x-direction relative to a frame $\mathcal{S}$. The nose of the ship passes the point $x = 0$ (in $\mathcal{S}$) at time $t = 0$, and at this event a light signal is sent from the nose of the ship to the rear. (a) Draw a spacetime diagram showing the worldlines of the nose and rear of the ship and the photon in $\mathcal{S}$. (b) When does the signal get to the rear of the ship in $\mathcal{S}$? (c) When does the rear of the ship pass $x = 0$ in $\mathcal{S}$? (After French 1966.)*

● **Problem 3–4:** *At noon a rocket ship passes the Earth at speed $\beta = 0.8$. Observers on the ship and on Earth agree that it is noon. Answer the following questions, and draw complete spacetime diagrams in both the Earth and rocket ship frames, showing all events and worldlines: (a) At 12:30 p.m., as read by a rocket ship clock, the*
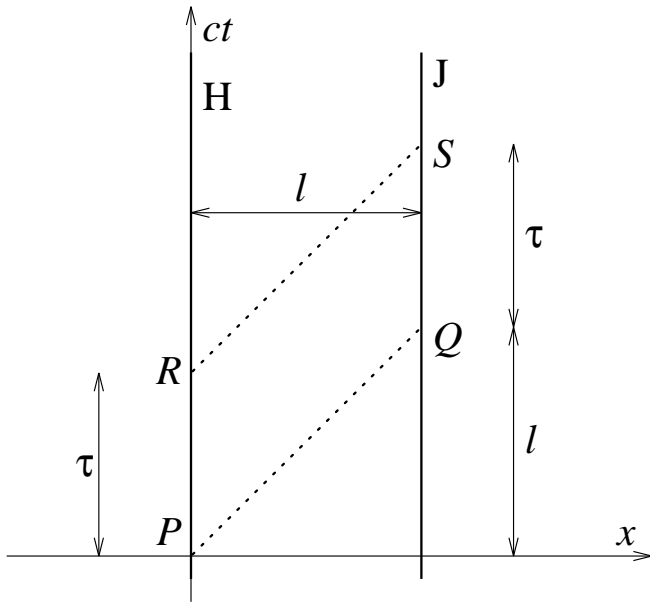
Figure 3.3: Spacetime diagram with worldlines of H, J, and the radio messages (dotted), along with the sending and receiving events. This diagram is drawn in H's rest frame; her worldline is vertical.
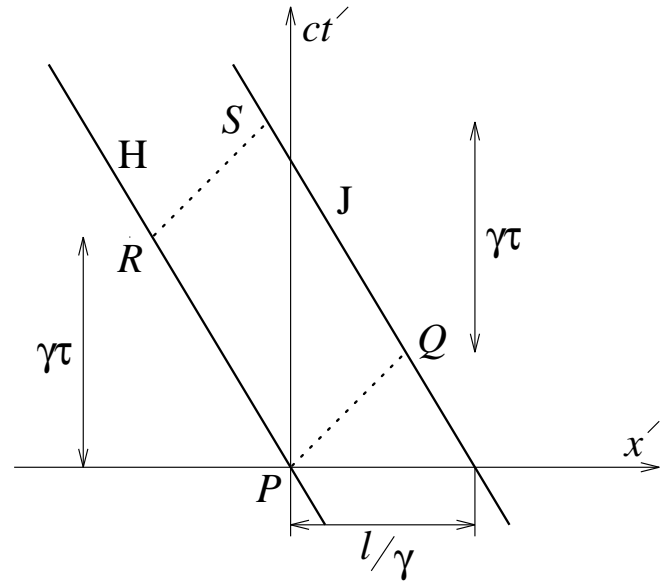

Figure 3.4: Spacetime diagram with worldlines of H, J, and the radio messages along with the sending and receiving events, now drawn in K's rest frame. Note the time dilation and length contraction.

ship passes an interplanetary navigational station that is fixed relative to the Earth and whose clocks read Earth time. What time is it at the station? (b) How far from Earth, in Earth coordinates, is the station? (c) At 12:30 p.m. rocket time, the ship reports by radio back to Earth. When does Earth receive this signal (in Earth time)? (d) The station replies immediately. When does the rocket receive the response (in rocket time)? (After French 1966.)

3.3 The "ladder and barn" paradox

Farmers Nettie (N) and Peter (P) own a barn of length ℓ and a ladder of length 2ℓ. They want to put the ladder into the barn, but of course it is too long. N suggests that P run with the ladder at speed $u = 0.866c$. At this speed $\gamma = 2$, so the ladder will be shortened by enough to fit into the barn. P objects. P argues that if he is running with the ladder, in *his* frame the ladder will still have length 2ℓ while the *barn* will be shortened to length $\ell/2$. The running plan will only make the problem worse!

They cannot both be right. Imagine P running with the ladder through the front door of the barn and out the back door, and imagine that the barn is specially equipped with a front door that closes immediately when the back of the ladder enters the barn (event C), and a back door that opens immediately when the front of the ladder reaches it (event D). Either there is a time when both doors are closed and the ladder is enclosed by the barn, or there is not. If there is such a time, we will say that the ladder fits, and if there is not, we will say that it does not fit. Who is right? Is N right that the ladder

is shorter and it will fit in the barn, or is P right that it isn't and won't?

If we draw spacetime diagrams of the ladder and barn in both frames we get Figure 3.5, where the front and back of the barn are labeled G and H respectively and the front and back of the ladder are J and K respectively. In N's frame, indeed, events C and D are simultaneous,
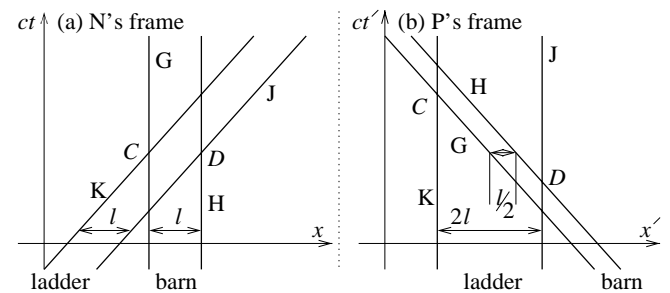

Figure 3.5: Worldlines of the front and back of the barn (G and H) and the front and back of the ladder (J and K) and events C and D in the rest frames of (a) N and (b) P. While events C and D are simultaneous in N's frame, they are not in P's.

so there is a brief time at which the ladder fits inside the barn. In P's frame, strangely enough, the events are no longer simultaneous! Event D happens long before event C, so there is no time at which the ladder is entirely inside the barn. So indeed both N and P are correct: whether or not the ladder fits inside the barn is a frame-dependent question; it depends on whether or not two events are simultaneous, and *simultaneity is relative.*

3.4   Relativity of simultaneity

How can we synchronize two clocks that are at rest with respect to one another but separated by a distance ℓ? The simplest thing to do is to put a lightbulb halfway between the two clocks, flash it, and have each clock start ticking when it detects the flash. The spacetime diagram in the rest frame $\mathcal{S}$ for this synchronizing procedure is shown in Figure 3.6, with the light bulb at the origin and the two clocks at $x = \pm\ell/2$. The flash is marked as event F and the detections of the flash as events G and H. Thereafter, the clock ticks are shown as marks on the clock worldlines. Simultaneous ticks lie on horizontal lines on the spacetime

Figure 3.7: The clocks as observed in frame $\mathcal{S}'$ along with events F, G, H, and the subsequent ticks. Although the clocks are synchronized in $\mathcal{S}$ they are not in $\mathcal{S}'$. Note that the lines of simultaneity (horizontal in $\mathcal{S}$) are slanted in $\mathcal{S}'$.

A simple thought experiment to demonstrate this consists of two clocks, synchronized and at rest in $\mathcal{S}$, exchanging photons simultaneously in $\mathcal{S}$, as shown in Figure 3.8. In
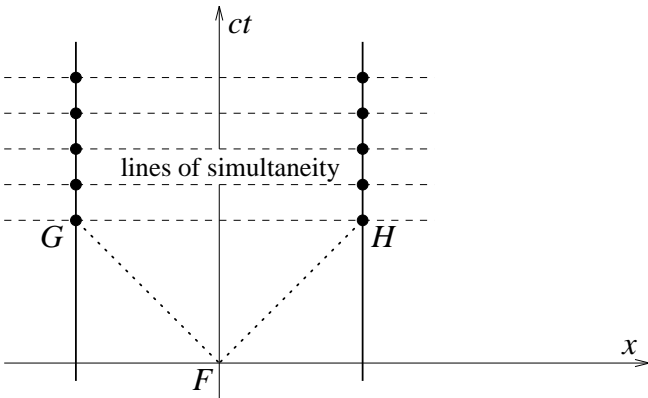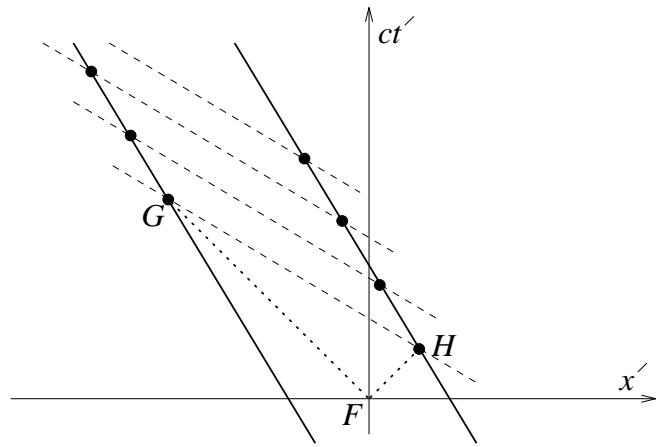

Figure 3.6: Synchronizing clocks at rest in frame $\mathcal{S}$ by flashing a lightbulb halfway between them at event F and having each clock start when it detects the flash (events G and H). After the two clocks receive the flashes, they tick as shown. Lines of simultaneity connect corresponding ticks and are horizontal.

diagram, because they occur at the same value of the time coordinate. In fact, the horizontal lines can be drawn in; they are *lines of simultaneity.*

Now consider a new frame $\mathcal{S}'$ which is moving at speed $+u = \beta\,c$ in the x-direction with respect to $\mathcal{S}$. In this new frame, the worldlines of the clocks are no longer vertical because they are moving at speed $-u$, but by Einstein's principle of relativity the flashes of light must still travel on 45° worldlines. So the spacetime diagram in $\mathcal{S}'$ looks like Figure 3.7. Note that in $\mathcal{S}'$ the lines of simultaneity joining the corresponding ticks of the two clocks are no longer horizontal. What does this mean? It means that two events which are simultaneous in $\mathcal{S}$ will not in general be simultaneous in $\mathcal{S}'$.

3.5   The boost transformation

We have seen in the previous section that "horizontal" lines of simultaneity in one frame become "tilted" in another frame moving with respect to the first, but can we quantify this? We can, and it turns out that the lines of simultaneity in frame $\mathcal{S}$ acquire slope $-\beta$ in frame $\mathcal{S}'$ (which moves at speed $+\beta\,c$ with respect to $\mathcal{S}$) just as the lines of constant position in $\mathcal{S}$ acquire slope $-1/\beta$ in $\mathcal{S}'$.
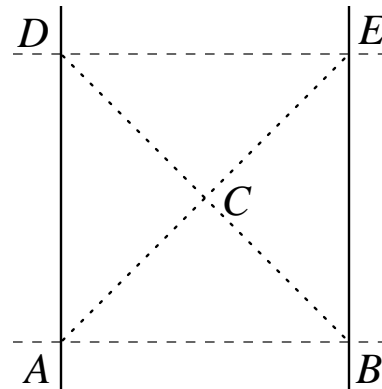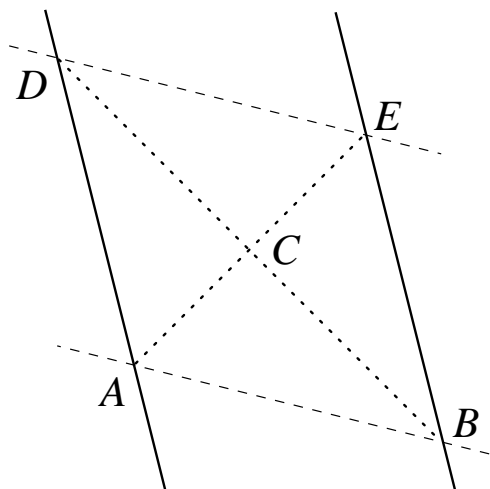

Figure 3.8: Clocks at rest and synchronized in frame $\mathcal{S}$ exchanging photons. They emit photons simultaneously at events A and B, the photons cross paths at event C, and then are received simultaneously at events D and E.

$\mathcal{S}$ they emit photons simultaneously at events A and B; the photons cross paths at event C; and then are received simultaneously at events D and E. In $\mathcal{S}'$ events A and B are no longer simultaneous, nor are events D and E. However, light must still travel on 45° worldlines and the photons must still cross at an event C halfway between the clocks. So the spacetime diagram in $\mathcal{S}'$ must look like Figure 3.9, with the square $ABED$ in $\mathcal{S}$ sheared into a parallelogram, preserving the diagonals as 45° lines. We know that the slope of the lines of constant position transform to lines of slope $-1/\beta$; in order to have the diagonals be 45° lines, we need the lines of simultaneity to transform to lines of slope $-\beta$.

This is really the essence of the boost transformation, the transformation from one frame to another moving with respect to it: the transformation is a shear or

Figure 3.9: Same as Figure 3.8 but in frame $\mathcal{S}'$.

Figure 3.10: Spacetime diagrams in frames (a) $\mathcal{S}'$ and (b) $\mathcal{S}$, each showing the time and space axes of both frames.

"crunch" along 45° lines. A shear is a linear transformation that does not involve rotation, but "squashes" coordinates along one direction, allowing them to expand along the perpendicular direction. In this case, these directions are photon trajectories or 45° worldlines.[§] We will derive the symbolic form of the boost transformation in Chapter 4, but for now these geometrical facts are all we need.
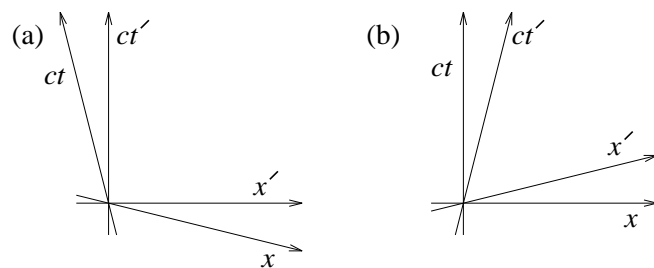
• **Problem 3–5:** *Prove, using whatever you need (including possibly Figures 3.8 and 3.9), that if the clock world lines have slope $1/\beta$ in some frame, the lines of simultaneity will have slope β. The shorter the proof, the better.*

3.6   Transforming space and time axes

One extremely useful way of representing the boost transformation between two frames on spacetime diagrams is to plot the space and time axes of both frames on both diagrams. This requires us to utilize two trivial facts: (a) the spatial axis of a frame is just the line of simultaneity of that frame which passes through the origin event $(x, ct) = (0, 0)$ and (b) the time axis is just the line of constant position which passes through $(0, 0)$. So if we (arbitrarily) identify origin events in the two frames,[¶] we can plot, in frame $\mathcal{S}'$, in addition to the x' and ct' axes, the locations of the x and ct axes of frame $\mathcal{S}$ (Figure 3.10(a)). We can also plot both sets of axes in frame $\mathcal{S}$. This requires boosting not by speed $+\beta c$ but rather by $-\beta c$ and, as you have undoubtedly figured out, this slopes the lines in the opposite way, and we get Figure 3.10(b). Again we see that the transformation is a shear. Note that the boost transformation is not a rotation, at least not in the traditional sense of the word!

We are now in a position to answer the question posed at the end of Section 2.1: How can it be that two observers, moving relative to one another, can *both* observe the other's clock to tick more slowly than their own? Imagine that observers at rest in $\mathcal{S}$ and $\mathcal{S}'$ both draw lines of constant position separated by 1 m of distance and lines of simultaneity separated by 1 m of time (3.3 ns) through the spacetime maps of their frames. In $\mathcal{S}$, the $\mathcal{S}$-observer's lines of constant position are vertical, and lines of simultaneity are horizontal. The $\mathcal{S}'$-observer's lines of constant position have slope $1/\beta$ and lines of simultaneity have slope $-\beta$, as seen in Figure 3.11. simultaneity. In $\mathcal{S}$, the horiztonal distance between the $\mathcal{S}'$-observer's lines of constant position is $(1 \text{ m})/\gamma$. Look carefully at Figure 3.11, which shows the ticks of each observer's clock along a line of constant position. If we travel along the $\mathcal{S}'$-observer's line of constant position, we find that we encounter ticks of the $\mathcal{S}'$ clock less frequently than lines of simultaneity in $\mathcal{S}$. On the other hand, if we travel along the $\mathcal{S}$-observer's line of constant position, we find that we also encounter ticks of the $\mathcal{S}$ clock less freqently than lines of simultaneity in $\mathcal{S}'$. That is, *both* observers find that the other's clock is going slow. There is no contradiction.

This point is subtle enough and important enough that the reader is advised to stare at Figure 3.11 until it is understood.
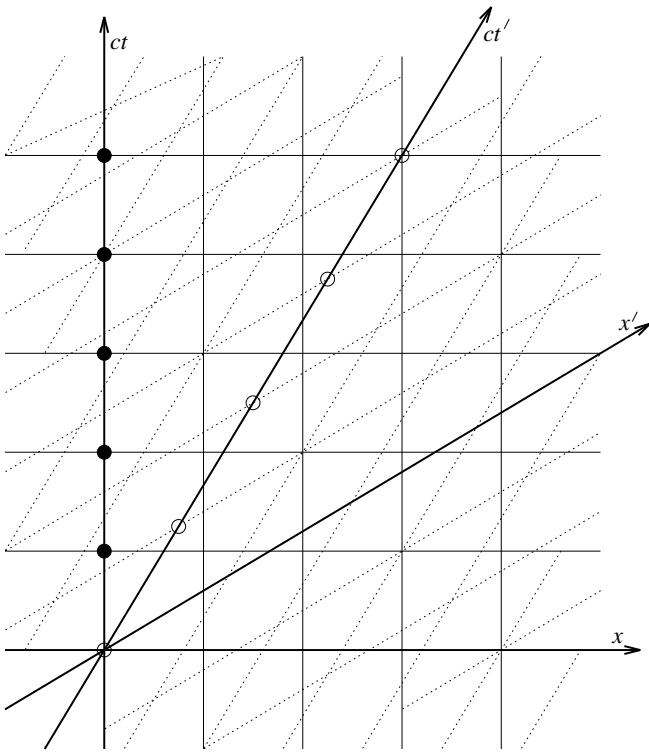

[§]The directions along which the squash and expansion take place are the *eigenvectors* of the transformation. The ambitious reader is invited to calculate the two corresponding eigenvalues.

[¶]The zero of time and space are arbitrary, so, with no loss of generality, we can assign these values so that the origin events coincide.

Figure 3.11: Spacetime diagram in frame $\mathcal{S}$, showing the spacetime grids drawn by the observer at rest in $\mathcal{S}$ (solid) and the observer at rest in $\mathcal{S}'$ (dotted). The $\mathcal{S}$-observer's clock ticks with solid dots and the $\mathcal{S}'$-observer's with open dots. Note that when travelling along a dotted line of constant position, clock ticks are encountered less frequently than solid lines of simultaneity and when travelling along a solid line of constant position, clock ticks are encountered less frequently than dotted lines of simultaneity. This explains how both observers can observe the other's clock to run slow.

Chapter 4

The Lorentz transformation

In this Chapter the invariant interval is introduced and the Lorentz transformation is derived and discussed. There is a lot of algebra but it is straightforward and the results are simple. The "twin paradox" is explained in terms of geodesics.

4.1 Proper time and the invariant interval

In 3-dimensional space, two different observers can set up different coordinate systems, so they will not in general assign the same coordinates to a pair of points P_1 and P_2. However they will agree on the *distance* between them. If one observer measures coordinate differences Δx, Δy and Δz between points A and B, and another, with a different coordinate system, measures $\Delta x'$, $\Delta y'$ and $\Delta z'$, they will both agree on the total distance Δr, defined by

$$
\begin{aligned}
(\Delta r)^2 &\equiv (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\
&= (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 .
\end{aligned} \quad (4.1)
$$

We would like to find a similar quantity for pairs of events: some kind of 'length' in 3+1-dimensional spacetime that is frame-independent, or the same for all observers. There is such a quantity, and it is called the *invariant interval* or simply *interval,* it is symbolized by $(\Delta s)^2$ and defined by

$$
\begin{aligned}
(\Delta s)^2 &\equiv (c\,\Delta t)^2 - (\Delta r)^2 \\
&\equiv (c\,\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 ,
\end{aligned} \quad (4.2)
$$

where Δt is the difference in time between the events, and Δr is the difference in space or the distance between the places of occurence of the events.

To demonstrate this, recall Section 2.1 in which we considered the flashes of a lightclock carried by D. In D's frame the flashes are separated by time $c\Delta t = 1$ m and distance $\Delta x = 0$. The interval between flashes is therefore $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 = 1$ m^2. In E's frame $c\Delta t' = \gamma\,(1\text{ m})$ and $\Delta x = \gamma\,u\,(1\text{ m})/c$, so the interval is $(\Delta s')^2 = \gamma^2\,(1 - u^2/c^2)\,(1\text{ m}^2)$. Since $\gamma \equiv (1 - u^2)^{-1/2}$, $(\Delta s')^2 = (\Delta s)^2 = 1$ m^2. Any other observer moving at any other speed w with respect to D will measure different time and space separations, but a similar argument will show that the interval is still 1 m^2.

The *proper time* $\Delta\tau$ between two events is the time experienced by an observer in whose frame the events take place at the same point if there is such a frame. As the above example shows, the square root of the invariant interval between the two events is c times the proper time, or $c\,\Delta\tau = \sqrt{(\Delta s)^2}$. The proper time is the length of time separating the events in D's frame, a frame in which both events occur at the same place. If the interval is positive, there always is such a frame, because positive interval means $|c\,\Delta t| > |\Delta r|$ so a frame moving at vector velocity $\boldsymbol{v} = (\Delta\boldsymbol{r})/(\Delta t)$, in which the events take place at the same point, is moving at a speed less than that of light.

If the interval between two events is less than zero, i.e., $(\Delta s)^2 < 0$, it is still invarant even though there is no frame in which both events take place at the same point. There is no such frame because necessarily it would have to move faster than the speed of light. To demonstrate the invariance in this case, consider the clock-synchronizing procedure described in Section 3.4: two flashes are emitted together from a point halfway between the clocks, separated by one meter. The clocks start when the flashes arrive, two events which are simultaneous in their rest frame. In the rest frame the two starting events are separated by $c\,\Delta t = 0$ and $\Delta x = 1$ m. The interval is $(\Delta s)^2 = -1$ m^2. In the frame moving at speed u with respect to the rest frame, the clocks are separated by $(1\text{ m})/\gamma$ and they are moving so the light takes time $(0.5\text{ m})/[\gamma(c+u)]$ to get to one clock and $(0.5\text{ m})/[\gamma(c-u)]$ to get to the other so $c\,\Delta t'$ is c times the difference between these, or $\gamma\,u\,(1\text{ m})/c$. Light travels at c so the displacement $\Delta x'$ is the c times the sum, or $\gamma\,(1\text{ m})$. The interval is $(\Delta s')^2 = -1$ m^2, same as in the rest frame. Since any other relative speed w could have been used, this shows that the interval is invariant even if it is negative.

Sometimes the *proper distance* $\Delta\lambda$ is defined to be the distance separating two events in the frame in which they occur at the same time. It only makes sense if the interval is negative, and it is related to the interval by $\Delta\lambda = \sqrt{|(\Delta s)^2|}$.

Of course the interval $(\Delta s)^2$ can also be exactly equal to zero. This is the case in which $(c\,\Delta t)^2 = (\Delta r)^2$, or in which the two events lie on the worldline of a photon. Because the speed of light is the same in all frames, a interval equal to zero in one frame must equal zero in all frames. Intervals with $(\Delta s)^2 = 0$ are called "lightlike" or "null" while those with $(\Delta s)^2 > 0$ are called "timelike"

and $(\Delta s)^2 < 0$ are called "spacelike". They have different causal properties, which will be discussed in Chapter 5.

4.2   Derivation of the Lorentz transformation

It would be nice to have algebraic formulae which allow us to compute the coordinates (ct', x', y', z') of an event in one frame given the coordinates (ct, x, y, z) of the event in some other frame. In this section we derive these formulae by assuming that the interval is invariant and asking "what kind of boost transformation will preserve the interval?", making one or two appeals to common sense on the way.

We want to find the linear* transformation that takes the coordinates (ct, x, y, z) of a 4-displacement in frame $\mathcal{F}$ to the coordinates (ct', x', y', z') it has in frame $\mathcal{G}$ so that the interval is invariant and $\mathcal{G}$ is moving at speed $u = \beta c$ in the x-direction with respect to $\mathcal{F}$.

In Section 2.3, we argued that there are no length distortions in the directions perpendicular to the direction of motion. This means that the y- and z-coordinates of an event in $\mathcal{F}$ must be the same as those in $\mathcal{G}$;

$$\begin{aligned} y' &= y \\ z' &= z \end{aligned} \tag{4.3}$$

Linearity requires that the x' and t' components must be given by

$$\begin{aligned} ct' &= L_{t't}\, ct + L_{t'x}\, x \\ x' &= L_{x't}\, ct + L_{x'x}\, x \end{aligned} \tag{4.4}$$

where the $L_{i'j}$ are constants; or, in matrix† form,

$$\left(\begin{array}{c} ct' \\ x' \end{array} \right) = \left(\begin{array}{cc} L_{t't} & L_{t'x} \\ L_{x't} & L_{x'x} \end{array} \right) \left(\begin{array}{c} ct \\ x \end{array} \right) \tag{4.5}$$

From the previous chapter, we know that two events that occur in $\mathcal{F}$ at the same place (so $\Delta x = 0$) but separated by time $c\,\Delta t$ occur in $\mathcal{G}$ separated by time $c\,\Delta t' = \gamma\, c\,\Delta t$ and therefore separated in space by $\Delta x' = -\beta\, c\,\Delta t' = -\beta\,\gamma\, c\,\Delta t$, where, as usual $\gamma \equiv (1 - \beta^2)^{-1/2}$. This implies

$$\begin{aligned} L_{t't} &= \gamma \\ L_{x't} &= \gamma\,\beta \end{aligned} \tag{4.6}$$

---

*The reader may ask: why need the transformation be linear? It needs to be linear because straight worldlines (i.e. constant-velocity worldlines) in one frame must transform into straight worldlines in all other frames.

†For a review of matrix algebra, see the excellent textbook by Strang (1976). In short, a column vector multiplied by a matrix makes another column vector according to the rule

$$\left(\begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} \right) = \left(\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array} \right) \left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right)$$

$$= \left(\begin{array}{c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \end{array} \right)$$

This is easily generalized to larger or smaller dimensions.

We also know that between any two events, the interval Δs^2 is the same in all frames. When $\Delta y = \Delta z = 0$, $(\Delta s)^2 = (c\,\Delta t)^2 - (\Delta x)^2$. Combined with the above two matrix elements, the requirement that $(\Delta s)^2 = (\Delta s)'^2$ implies

$$\begin{aligned} L_{t'x} &= -\gamma\,\beta \\ L_{x'x} &= \gamma \end{aligned} \tag{4.7}$$

So we find that the transformation of the coordinates from one frame $\mathcal{F}$ to another $\mathcal{G}$ that is moving in the x-direction at relative speed $+u = \beta\, c$ is given by

$$\left(\begin{array}{c} ct' \\ x' \\ y' \\ z' \end{array} \right) = \left(\begin{array}{cccc} \gamma & -\gamma\,\beta & 0 & 0 \\ -\gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \left(\begin{array}{c} ct \\ x \\ y \\ z \end{array} \right) \tag{4.8}$$

4.3   The Lorentz transformation

The Lorentz transformation (hereafter LT) is very important and deserves some discussion. The LT really transforms differences $(c\,\Delta t, \Delta x, \Delta y, \Delta z)$ between the coordinates of two events in one frame to differences $(c\,\Delta t', \Delta x', \Delta y', \Delta z')$ in another frame. This means that if one is going to apply the LT directly to event coordinates, one must be very careful that a single event is at the origin $(0, 0, 0, 0)$ of both frames.‡ In the previous section, we placed event P at the origin of both frames.

A simple consistency check we could apply to the LT is the following: If we boost to a frame moving at u, and then boost back by a speed $-u$, we should get what we started with. In other words, LTs with equal and opposite speeds should be the inverses of one another. If we change $u \to -u$, we have $\beta \to -\beta$ and $\gamma \to \gamma$, so boosting the coordinates (ct', x') in frame $\mathcal{K}$ back to $\mathcal{H}$ and giving the new coordinates double-primes, we have

$$\begin{aligned} ct'' &= \gamma\, ct' + \beta\,\gamma\, x' \\ &= \gamma\,(\gamma\, ct - \beta\,\gamma\, x) + \beta\,\gamma\,(-\beta\,\gamma\, ct + \gamma\, x) \\ &= \gamma^2\,(ct - \beta\, x - \beta^2\, ct + \beta\, x) \\ &= \gamma^2\,(1 - \beta^2)\, ct \\ &= ct \end{aligned} \tag{4.9}$$

$$\begin{aligned} x'' &= \beta\,\gamma\, ct' + \gamma\, x' \\ &= \beta\,\gamma\,(\gamma\, ct - \beta\,\gamma\, x) + \gamma\,(-\beta\,\gamma\, ct + \gamma\, x) \\ &= \gamma^2\,(\beta\, ct - \beta^2\, x - \beta\, ct + x) \\ &= \gamma^2\,(1 - \beta^2)\, x \\ &= x \;, \end{aligned} \tag{4.10}$$

so indeed, the boost of $-u$ is the inverse of the boost of u.

The LT as defined above has the primed frame $(\mathcal{K})$ moving at speed $+u$ with respect to the unprimed frame $(\mathcal{H})$. This is not a universal convention, but I will try to stick to it.

---

‡There is a more general class of transformations, *Poincaré transformations*, which allow translations of the coordinate origin as well LTs (which include boosts and, as we will see, rotations).

The group of all LTs includes all linear transformations that preserve the interval[§]. This means that LTs include space rotations with no boost, for example

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4.11)$$

LTs also include boosts in arbitrary directions, not just the x-direction. For an arbitrary relative velocity $\boldsymbol{u} = (u_x, u_y, y_z)$ of frame $\mathcal{S}'$ with respect to $\mathcal{S}$, the corresponding LT is

$$\begin{pmatrix} \gamma & -\gamma\,\beta_x & -\gamma\,\beta_y & -\gamma\,\beta_z \\ -\gamma\,\beta_x & 1 + \frac{(\gamma-1)\,\beta_x^2}{\beta^2} & \frac{(\gamma-1)\,\beta_x\,\beta_y}{\beta^2} & \frac{(\gamma-1)\,\beta_x\,\beta_z}{\beta^2} \\ -\gamma\,\beta_y & \frac{(\gamma-1)\,\beta_x\,\beta_y}{\beta^2} & 1 + \frac{(\gamma-1)\,\beta_y^2}{\beta^2} & \frac{(\gamma-1)\,\beta_y\,\beta_z}{\beta^2} \\ -\gamma\,\beta_z & \frac{(\gamma-1)\,\beta_x\,\beta_z}{\beta^2} & \frac{(\gamma-1)\,\beta_y\,\beta_z}{\beta^2} & 1 + \frac{(\gamma-1)\,\beta_z^2}{\beta^2} \end{pmatrix}$$
$$(4.12)$$

where we define

$$\beta_x \equiv u_x/c$$
$$\beta_y \equiv u_y/c$$
$$\beta_z \equiv u_z/c$$
$$\beta^2 \equiv \beta_x^2 + \beta_y^2 + \beta_z^2$$
$$\gamma \equiv (1 - \beta_x^2 - \beta_y^2 - \beta_z^2)^{-1/2} \qquad (4.13)$$

(see, e.g., Jackson, 1975, Chapter 11). And, of course, any composition of arbitrary LTs is also an LT.

• **Problem 4–1:** *Transform the events A $(ct, x) = (0,0)$, B $(0, 1\,\mathrm{m})$, C $(1/2\,\mathrm{m}, 1/2\,\mathrm{m})$, D $(1\,\mathrm{m}, 0)$, and E $(1\,\mathrm{m}, 1\,\mathrm{m})$ into a frame $\mathcal{S}'$ moving at speed $+0.6c$ in the x-direction with respect to the unprimed frame $\mathcal{S}$. Draw spacetime diagrams of both frames showing the five events.*

To check your answer: notice that A, C, and E all lie on a 45° worldline, as do B, C, and D. The LT must transform 45° worldlines to 45° worldlines because the speed of light is c in all frames.

• **Problem 4–2:** *Write down the transformation from a frame $\mathcal{S}$ to a frame $\mathcal{S}'$ moving at $+0.5c$ in the x-direction and then to another frame $\mathcal{S}''$ moving at $+0.5c$ in the x-direction relative to $\mathcal{S}'$. What is the complete transformation from $\mathcal{S}$ to $\mathcal{S}''$? What relative speed between frames $\mathcal{S}$ and $\mathcal{S}''$ does your answer imply?*

• **Problem 4–3:** *Show that the transformations given for a coordinate rotation and for a boost in an arbitrary direction preserve the interval.*

• **Problem 4–4:** *Do space reflections and time-reversals preserve the interval?*

• **Problem 4–5:** *Denote by E the event on the ct-axis of a spacetime diagram that is a proper time $c\tau$ from the origin. What is the locus of all events on the spacetime diagram that are separated from the origin by the same proper time?*

The answer should be a hyperbola that asymptotes to the line $ct = x$ but which is horizontal on the spacetime diagram right at E.

• **Problem 4–6:** *Denote by F the event on the x-axis of a spacetime diagram that is a distance ℓ from the origin. What is the locus of all events which are separated from the origin by the same interval as F?*

4.4 Velocity addition

We are now in a position to derive the correct velocity addition law that replaces the simple but incorrect one suggested in Section 1.2: If A moves at speed $+u$ in the x-direction with respect to B, and A throws a cantaloupe at speed $+v$ in the x-direction relative to himself, at what speed w does B observe the cantaloupe to travel? The simple but incorrect answer is $w = u + v$. The correct answer can be quickly calculated with a Lorentz transformation. Call the throwing event T and put it at the origin of both frames, so $(ct_T, x_T) = (ct'_T, x'_T) = (0, 0)$, where A's frame gets the primes. Now imagine that at some time t' later in A's frame, the cantaloupe explodes, this explosion event E must occur at coordinates (ct', vt') in A's frame. In B's frame, by definition, T occurs at the origin, but by applying the LT with speed $-u$ (defining $\beta \equiv u/c$ and γ accordingly) E now occurs at

$$\begin{aligned} ct &= \gamma\,ct' + \beta\,\gamma\,vt' \\ x &= \beta\,\gamma\,ct' + \gamma\,vt' \end{aligned} \qquad (4.14)$$

The speed w measured by B is simply x/t or

$$\begin{aligned} w &= c\,\frac{\beta\,\gamma\,ct' + \gamma\,vt'}{\gamma\,ct' + \beta\,\gamma\,vt'} \\ &= \frac{u + v}{1 + uv/c^2} \end{aligned} \qquad (4.15)$$

which is less than $u + v$. Spacetime diagrams for this calculation are shown in Figure 4.1.
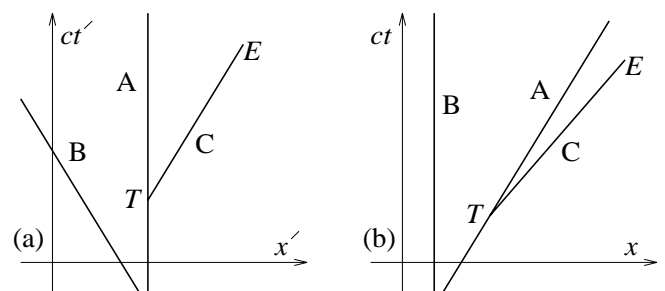

Figure 4.1: Spacetime diagrams of the throw T and explosion E of C by A, as observed by (a) A and (b) B for the purposes of computing the velocity addition law.

[§]In fact, the astute reader will notice that there are linear transformations which preserve the interval but involve reversing the direction of time or reflecting space through a plane. These do indeed satisfy the criteria to be LTs but they are known as "improper" LTs because they do not correspond to physically realizable boosts. On the other hand, they do have some theoretical meaning in relativistic quantum mechanics, apparently.

• **Problem 4–7:** *In an interplanetary race, slow team X is travelling in their old rocket at speed* 0.9*c relative to the finish line. They are passed by faster team Y, observing Y to pass X at* 0.9*c. But team Y observes fastest team Z to pass Y's own rocket at* 0.9*c. What are the speeds of teams X, Y and Z relative to the finish line?*

The answer is not 0.9*c*, 1.8*c*, and 2.7*c*!

• **Problem 4–8:** *An unstable particle at rest in the lab frame splits into two identical pieces, which fly apart in opposite directions at Lorentz factor* $\gamma = 100$ *relative to the lab frame. What is one particle's Lorentz factor relative to the other? What is its speed relative to the other, expressed in the form* $\beta \equiv 1 - \epsilon$?

• **Problem 4–9:** *Determine the transformation law for an arbitrary 3-vector velocity* $\boldsymbol{v} = (v_x, v_y, v_z)$.

4.5   The twin paradox

Lin (L) and Ming (M) are twins, born at the same time, but with very different genes: L is an astronaut who likes to explore outer space, and M is a homebody who likes to stay at home on Earth and read novels.¶ When both L and M turn 20, L leaves on a journey to a nearby star. The star is $\ell = 30$ light years away and L chooses to travel out at speed $u = 0.99c$ and then immediately turn around and come back. From M's point of view, the journey will take time $T = 2\ell/u \approx 60$ yr, so L will return when M is 80. How much will L have aged over the same period?

In Section 2.1 we learned that moving clocks go slow, so L will have aged by $T' = T/\gamma$, where $\gamma \equiv (1 - \beta^2)^{-1/2}$ and $\beta \equiv u/c$. For $u = 0.99c$, $\gamma = 7$, so L will have aged less than 9 yr. That is, on L's arrival home, M will be 80, but L will only be 28! Strange, but in this special relativistic world, we are learning to live with strangeness.

During his journey, Ming starts to get confused about this argument. After all, there is no preferred reference frame. If one looks at the Earth from the point of view of Ming's rocket, one sees the Earth travel out at speed u and come back. So isn't L's clock the one that runs slow, and won't L the one who will be younger upon return? How can this be resolved?

In Figure 4.2, the worldlines of L and M are plotted in the rest frame of the Earth (frame $\mathcal{S}$), with L's departure marked as event D, L's turnaround at the distant star as T and her return home as R. You will recall that in Section 4.1 we saw that along a worldline, the proper time, or time elapsed for an observer travelling along the worldline, is the square root of the interval $(\Delta s)^2 = (c\,\Delta t)^2 - (\Delta x)^2$. M does not move, so $\Delta x = 0$ and the proper time for him is just Δt_{DR}. L moves very quickly, so (Δx) is not zero, so her proper time out to event T and back again will be much smaller than simply Δt_{DR}. Smaller, of course, by a factor $1/\gamma$.

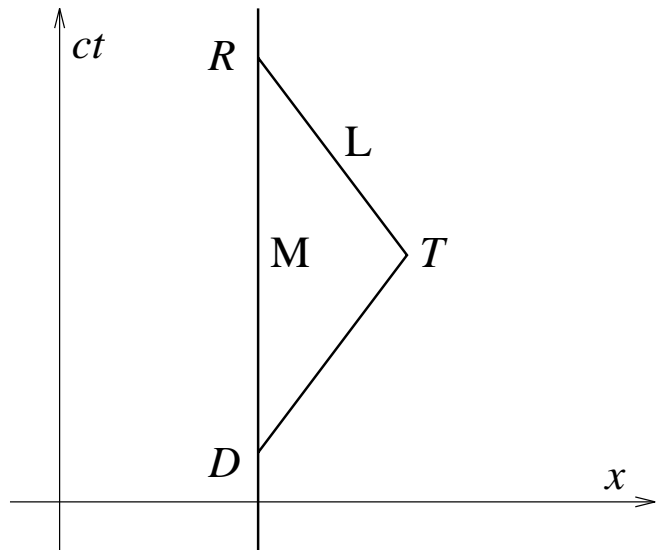Let's draw this now in L's frame. But we have a problem: just what frame do we choose? Do we choose the

Figure 4.2: Worldlines of the twins L and M in frame $\mathcal{S}$, with L's departure marked as D, turnaround as T and return home as R.

frame $\mathcal{S}'$ that is L's rest frame on her way out to the star, or the frame $\mathcal{S}''$ that is L's rest frame on the way back? We cannot choose both because they are different frames: L *changes frames* at event T. This breaks the symmetry and resolves the paradox: M travels from event D to event R in a single frame with no changes, while L changes frames. L's worldline is crooked while M's is straight‖.

It is easy to show that given any two events and a set of worldlines that join them, the worldline corresponding to the path of longest proper time is the straight line. Just as in Euclidean space the straight line can be defined as the shortest path between two points, in spacetime the straight worldline can be defined as the path of *longest proper time*. This is in fact the definition, and straight worldlines are called *geodesics*.

• **Problem 4–10:** *Prove that the straight worldline joining any two events E and F is the line of maximum proper time. Hint: begin by transforming into the frame in which E and F occur at the same place.*

• **Problem 4–11:** *Imagine that every year, on their respective birthdays, each twin sends the other a radio message (at the speed of light). Re-draw Figure 4.2 on graph paper and draw, as accurately as possible, L's birthday messages in red and M's birthday messages in blue. How many messages does each twin receive? At what ages to*

¶Do not regard this statement as a position on the nature/nurture debate.

‖Another, fundamentally incorrect, but nonetheless useful, way to distinguish the twins is to imagine that despite their genetic differences, they are both avid coffee drinkers. If they each spend the entire time between events D and R drinking coffee, L experiences no trouble at all, but M finds that he spills his coffee all over himself at event T. After all, his spaceship suffers a huge acceleration at that time. L experiences no such trauma. This explanation is fundamentally flawed because if we allow for gravitational forces, there are many ways to construct twin paradoxes which do not involve this asymmetry.

they receive them?

- **Problem 4–12:** *Imagine that rather than taking one long trip out and back, Ming in fact takes five shorter trips out and back, but all at the same speed β, and elapsing the same total time (on Lin's clock) for all the trips, as in the single-trip case. What effect does this have on Ming's aging relative to Lin's, as compared with the single-trip case? Estimate how much less a commercial airline pilot ages relative to her or his spouse over her or his lifetime.*

Chapter 5

Causality and the interval

The sign of the interval $(\Delta s)^2$ (i.e., whether it is positive or negative) is discussed in terms of causality in this Chapter. If one event can affect another causally, the interval between them must be positive. By preserving the interval, therefore, the Lorentz transformation preserves also the causal structure of the Universe, provided that nothing travels faster than light. This is the reason for that universal speed limit.

5.1   The ladder and barn revisited

Recall the "ladder and barn" paradox discussed in Section 3.3, in which N is at rest with respect to a barn, and P is carrying a long ladder but running so that it will be length contracted and therefore fit.

Confused by the discussion of relativity of simultaneity in Chapter 3, N decides to *prove* that ladder does indeed fit into the barn by replacing the back door with an incredibly strong, rigid, and heavy back wall that does not open. Now when P enters the barn, he cannot leave, and the question is: does the front door ever close at all? If it closes, the ladder *must* be really inside the barn in *all* frames because there is no back door through which it can be exiting. Thus instead of asking whether event C happens before or after D, a frame-dependent question, we are asking whether C happens at all. This is a frame-independent question.*

In N's frame, event C, the closing of the front door, *must* happen because the front of the ladder does not hit the back wall until event C has occurred. That is, the ladder does not even "know" that the back door has been replaced by a brick wall until event C has occurred, so if event C, the closing of the front door, happened when the back door was open, it must still happen now that the back door is no longer there.

In P's frame the front of the ladder hits the back of the barn before the back of the ladder enters, as we saw in Section 3.3. But does this mean that the ladder will stop and event C will no longer happen? To answer this question, we will have to actually do some *Physics* for the

---

*Events are frame-independent entities in the sense that if an event occurs in one frame, it must occur in all. One cannot "undo" the fact that one sneezed by changing frames! On the other hand, relationships between events such as simultaneity are frame-dependent or relative.

first time in these notes.

If I am standing at one end of a long table of length ℓ and I push on the table to move it, how quickly can someone standing at the other end feel the table move? My pushing on the table sets up a compression wave that travels at the speed of sound c_s in the table. The person at the other end feels the push when the wave gets there, at a time ℓ/c_s after I push. In everyday experience, this time is fairly short, so we are not aware of the time delay between the push at one end and the feeling at the other. But if we stand at opposite ends of a very long, stretched slinky, this time delay is easily observable.

Because, as we will see, no object or piece of matter can ever travel faster than the speed of light and because all information is transferred via either matter or light itself, no information or signal or, in particular, compression wave, can ever travel faster than the speed of light. This means that no matter how rigid and strong I build my table, the earliest possible time that the person at the other end can feel my push is at a time ℓ/c after I push, where c is now the speed of light.

Why this digression? Because it applies to the problem at hand. Sure, in P's frame, the front of the ladder hits the back of the barn before the back of the ladder enters, but this information cannot reach the back of the ladder until some finite time after the collision. So the back of the ladder doesn't know that anything has gone awry at the front and it continues to move. When does the back of the ladder learn of the front's collision? To answer this we need to draw spacetime diagrams. Figure 5.1 shows the spacetime diagrams in the two frames. Event D is the collision of the ladder with the back wall, and we have added event E, the earliest possible moment at which the back of the ladder can learn of the collision at the front. This event is separated from the collision event by a photon trajectory, because the maximum speed at which the information can travel is the speed of light. In both frames we see that the back of the ladder enters the barn and event C occurs before the back of the ladder learns about the collision. In other words, the back of the ladder makes it into the barn and the door closes behind it. What does this imply? It implies that the ladder must be compressible or fragile. The fact that the speed of sound in the ladder cannot exceed the speed of light ensures that all materials are compressible.
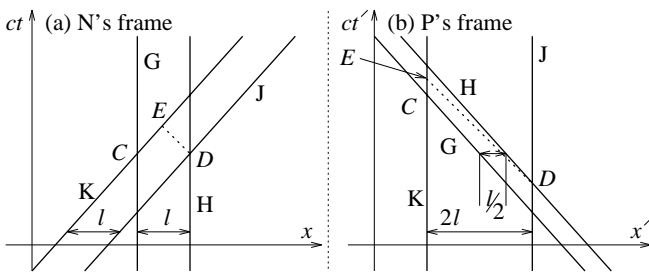
Figure 5.1: Same as Figure 3.5 but now event D is a collision rather than an exit. The news of the collision cannot travel faster than the speed of light so it cannot reach the back of the ladder before event E.

Loosely speaking, this is because a totally incompressible substance has an infinite sound speed, and that is not allowed. There are many fun problems in relativity based on this type of argument, discussion of which is prevented by lack of space. One important application is a proof that dark (i.e., not burning nuclear fuel), compact objects more massive than about three times the mass of the Sun must be black holes: any other material, even a crystal composed of pure neutrons, can only hold itself up under that kind of pressure if it is so rigid that the speed of sound in the material would necessarily exceed the speed of light!

• **Problem 5–1:** *Imagine a plank of length ℓ supported at both ends by sawhorses in a gravitational field of acceleration g. One support is kicked out. What is the minimum time the other end of the plank could "know" that the one end has lost its support? Roughly speaking, what distance Δy will the one end fall before the other can know? How much does the board bend, and, to order of magnitude, what does this tell you about, say, the Young's modulus of the board?*

• **Problem 5–2:** *Imagine a wheel of radius R consisting of an outer rim of length $2\pi R$ and a set of spokes of length R connected to a central hub. If the wheel spins so fast that its rim is travelling at a significant fraction of c, the rim ought to contract to less than $2\pi R$ in length by length contraction, but the spokes ought not change their lengths at all (since they move perpendicular to their lengths). How do you think this problem is resolved given the discussion in this Section? If you find a solution to this problem which does not make use of the concepts introduced in this Section, come see me right away!*

5.2   Causality

Event order is relative, but it is subject to certain constraints. By changing frames in the ladder-and-barn paradox, we can make event D precede, be simultaneous with, or follow event E. But we cannot make *any* pair of events change their order simply by changing frames. For instance, if Quentin (Q) throws a ball to Rajesh (R), the event of the throw A must precede the event of the catch

B in all frames. After all, it is impossible for R to catch the ball before Q throws it!

If indeed events A and B are the throwing and catching of a ball, we can say something about their x and t coordinates. The spatial separation Δx between the events must be less than the time (in dimensions of distance) $c\,\Delta t$ between the events because the ball cannot travel faster than the speed of light. For such a pair of events the interval

$$(\Delta s)^2 = (c\,\Delta t)^2 - (\Delta x)^2 \qquad (5.1)$$

must be positive. Events with positive interval must occur in the same order in all frames because activity at the earlier event can affect activity at the later event. Such a pair of events has a *timelike* spacetime separation, and it is sometimes said that A is in the *causal history* of B, or B is in the *causal future* of A.

In the case of events C and D in the ladder-and-barn paradox, the interval between the events is negative, and any signal or information or matter traveling between the events would have to travel faster than the speed of light. Thus activity at each of these events is prevented from affecting activity at the other, so there is no logical or physical inconsistency in having a boost transformation change their order of occurence. Such a pair of events has a *spacelike* spacetime separation. They are *causally disconnected*.

For completeness we should consider events D and E in the ladder-and-barn paradox. These events are separated by a photon world line or $(c\,\Delta t)^2 = (\Delta x)^2$, so the interval is zero between these events. Such a pair of events is said to have a *lightlike* or *null* spacetime separation. Two events with a null separation in one frame must be have a null separation in all frames because the speed of light is the same in all frames.

5.3   Nothing can travel faster than the speed of light

The well-known speed limit—nothing can travel faster than the speed of light—follows from the invariant causal structure of the Universe. If one event is in the causal history of another in one frame, it must be in that causal history in all frames, otherwise we have to contend with some pretty wacky physics.[†] For instance, reconsider the above example of Q and R playing catch. Imagine that Q and R are separated by ℓ in their rest frame $\mathcal{S}$, and Q throws the ball to R at twice the speed of light. The spatial separation between events A and B is $\Delta x = \ell$ and the time separation is $c\Delta t = \ell/2$. Now switch to a frame $\mathcal{S}'$ moving at speed v in the direction pointing from Q to R. Applying the Lorentz transformation, in this frame

$$\Delta t' = \gamma\,\ell\left(\frac{1}{2} - v\right) \qquad (5.2)$$

[†]The reader who objects that special relativity is already fairly wacky will be ignored.

which is less than zero if $v > 1/2$. In other words, in frames $\mathcal{S}'$ with $v > 1/2$, event B precedes event A. I.e., in $\mathcal{S}'$ R must catch the ball before Q throws it. A little thought will show this to be absurd; we are protected from absurdity by the law that nothing, no object, signal or other information, can travel faster than the speed of light. Thus we have been justified, earlier in this chapter, and elsewhere in these notes, in assuming that nothing can travel faster than the speed of light.

Chapter 6

Relativistic mechanics

6.1 Scalars

A *scalar* is a quantity that is the same in all reference frames, or for all observers. It is an *invariant* number. For example, the interval $(\Delta s)^2$ separating two events A and B is a scalar because it is the same in all frames. Similarly, the proper time $\Delta\tau$ between two events on a worldline is a scalar. In Chapter 2, the number of ticks of D's clock in going from planet A to planet B is a scalar because although observers disagree on how far apart the ticks are in time, they agree on the total number.

It is worth emphasizing that the time interval Δt between two events, or the distance Δx between two events, or the the length ℓ separating two worldlines are *not* scalars: they do not have frame-independent values*.

6.2 4-vectors

Between any two distinct events A and B in spacetime, there is a time difference $c\,\Delta t$ and three coordinate difference Δx, Δy and Δz. These four numbers can be written as a vector $\vec{x}$ with four components, which is called a *4-vector*:

$$\vec{x} = (c\,\Delta t, \Delta x, \Delta y, \Delta z) \qquad (6.1)$$

The 4-vector† $\vec{x}$ is actually a frame-independent object, although this is a fairly subtle concept. The components of $\vec{x}$ are not frame-independent, because they transform by the Lorentz transformation (Section 4.3). But event A is frame independent: if it occurs in one frame, it must occur in all frames, and so is event B, so there is some frame-independent meaning to the 4-vector displacement or *4-displacement* between these events: it is the 3+1-dimensional arrow in spacetime that connects the two events.

The frame-independence can be illustrated with an analogy with 3-dimensional space. Different observers set up different coordinate systems and assign different coordinates to two points P and Q, say Pittsburgh, PA and Queens, NY. Although both observers agree that they are talking about Pittsburgh and Queens, they assign different coordinates to the points. The observers can

also discuss the 3-displacement $\boldsymbol{r}$ separating P and Q. Again, they may disagree on the coordinate values of this 3-vector, but they will agree that it is equal to the vector that separates points P and Q. They will also agree on the length of $\boldsymbol{r}$ and they will agree on the angle it makes with any other vector $\boldsymbol{s}$, say the vector displacement between P and R (Richmond, VA). In this sense the points and 3-vector are frame-independent or coordinate-free objects, and it is in the same sense that events and 4-vectors are frame-independent objects.

With each 4-displacement we can associate a scalar: the interval $(\Delta s)^2$ along the vector. The interval associated with $\vec{x}$ is

$$(\Delta s)^2 = (c\,\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \qquad (6.2)$$

Because of the similarity of this expression to that of the dot product between 3-vectors in three dimensions, we also denote this interval by a dot product and also by $|\vec{x}|^2$:

$$\vec{x} \cdot \vec{x} \equiv |\vec{x}|^2 \equiv (c\,\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \quad (6.3)$$

and we will sometimes refer to this as the *magnitude* or *length* of the 4-vector.

We can generalize this dot product to a dot product between any two 4-vectors $\vec{a} = (a_t, a_x, a_y, a_z)$ and $\vec{b} = (b_t, b_x, b_y, b_z)$:

$$\vec{a} \cdot \vec{b} \equiv a_t\,b_t - a_x\,b_x - a_y\,b_y - a_z\,b_z \qquad (6.4)$$

It is easy to show that this dot product obeys the rules we expect dot products to obey: associativity over addition and commutativity. The nice result is that the dot product produces a scalar. That is, the dot product of any two 4-vectors in one frame equals their dot product in any other frame.

When frames are changed, 4-displacements transform according to the Lorentz transformation. Because 4-displacements are 4-vectors, it follows that all 4-vectors transform according to the Lorentz transformation. This provides a simple (though slightly out-of-date) definition of a 4-vector: an ordered quadruple of numbers that transforms according to the Lorentz transformation.

Because scalars, by definition, do not change under a Lorentz transformation, any 4-component object which transforms according to the Lorentz transformation can

---

*Forget high school—where all single-component numbers were probably referred to as "scalars."

†The convention in these notes is to denote 4-vectors with vector hats and 3-vectors with bold face symbols.

be multiplied or divided by a scalar to give a new four-component object which also transforms according to the Lorentz transformation. In other words, a 4-vector multiplied or divided by a scalar is another 4-vector.

• **Problem 6–1:** *Show that the 3+1-dimensional dot product obeys associativity over addition, i.e., that*

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \qquad (6.5)$$

and commutativity, i.e., that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

• **Problem 6–2:** *Show that the dot product of two 4-vectors is a scalar. That is, show that for any two 4-vectors $\vec{a}$ and $\vec{b}$, their dot product in one frame $\mathcal{S}$ is equal to their dot product in another $\mathcal{S}'$ moving with respect to $\mathcal{S}$.*

• **Problem 6–3:** *Show that 4-vectors are closed under addition. That is, show that for any two 4-vectors $\vec{a}$ and $\vec{b}$, their sum $\vec{c} = \vec{a} + \vec{b}$ (i.e., each component of $\vec{c}$ is just the sum of the corresponding components of $\vec{a}$ and $\vec{b}$) is also a 4-vector. Show this by comparing what you get by Lorentz transforming and then summing with what you get by summing and then Lorentz transforming.*

6.3   4-velocity

What is the 3+1-dimensional analog of velocity? We want a 4-vector so we want a four-component object that transforms according to the Lorentz transformation. In 3-dimensional space, 3-velocity $\boldsymbol{v}$ is defined by

$$\boldsymbol{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{d\boldsymbol{r}}{dt} \qquad (6.6)$$

where Δt is the time it takes the object in question to go the 3-displacement $\Delta \boldsymbol{r}$. The naive 3+1-dimensional generalization would be to put the 4-displacement $\Delta \vec{x}$ in place of the 3-displacement $\Delta \boldsymbol{r}$. However, this in itself won't do, because we are dividing a 4-vector by a non-scalar (time intervals are not scalars); the quotient will not transform according to the Lorentz transformation. The fix is to replace Δt by the proper time $\Delta \tau$ corresponding to the interval of the 4-displacement; the 4-velocity $\vec{u}$ is then

$$\vec{u} \equiv \lim_{\Delta \tau \to 0} \frac{\Delta \vec{x}}{\Delta \tau} \qquad (6.7)$$

When we take the limit we get derivatives, and the proper time $\Delta \tau$ is related to the coordinate time Δt by $\gamma \Delta \tau = \Delta t$ (where, as usual, $\gamma \equiv (1 - \beta^2)^{-1/2}$ and $\beta \equiv |\boldsymbol{v}|/c$), so

$$\begin{aligned} \vec{u} &= \frac{d\vec{x}}{d\tau} \\ &= \left(c \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right) \\ &= \left(c\gamma \frac{dt}{dt}, \gamma \frac{dx}{dt}, \gamma \frac{dy}{dt}, \gamma \frac{dz}{dt} \right) \\ &= (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z) \qquad (6.8) \end{aligned}$$

where (v_x, v_y, v_z) are the components of the 3-velocity $\boldsymbol{v} = d\boldsymbol{r}/dt$. Although it is unpleasant to do so, we often write 4-vectors as two-component objects with the first component a single number and the second a 3-vector. In this notation

$$\vec{u} = (\gamma c, \gamma \boldsymbol{v}) \qquad (6.9)$$

What is the magnitude of $\vec{u}$? There are several ways to derive it, the most elegant is as follows. The magnitude $|\vec{u}|^2$ must be the same in all frames because $\vec{u}$ is a four-vector. Let us change into the frame in which the object in question is at rest. In this frame $\vec{u} = (c, 0, 0, 0)$ because $\boldsymbol{v} = (0, 0, 0)$ and $\gamma = 1$. Clearly in this frame $|\vec{u}|^2 = c^2$ or $|\vec{u}| = c$. It is a scalar so it must have this value in all frames. Thus $|\vec{u}| = c$ in all frames. This trivial "proof" is a good model for problem-solving in special relativity: identify something which is frame-independent, transform into a frame in which it is easy to calculate, and calculate it. The answer will be good for all frames.

The reader may find this a little strange. Some particles move quickly, some slowly, but for all particles, the magnitude of the 4-velocity is c. But this is not strange, because we need the magnitude to be a scalar, the same in all frames. If I change frames, some of the particles that were moving quickly before now move slowly, and some of them are stopped altogether. Speeds (magnitudes of 3-velocities) are relative; the magnitude of the 4-velocity has to be invariant.

• **Problem 6–4:** *Apply the formula for the magnitude of a 4-vector to the general 4-velocity $(\gamma c, \gamma v_x, \gamma v_y, \gamma v_z)$ to show that its magnitude is indeed c.*

6.4   4-momentum, rest mass and conservation laws

Just as in non-relativistic 3-space, where 3-momentum was defined as mass times 3-velocity, in spacetime 4-momentum $\vec{p}$ is mass m times 4-velocity $\vec{u}$. Under this definition, the mass must be a scalar if the 4-momentum is going to be a 4-vector. If you are old enough, you may have heard of a quantity called "relativistic mass" which increases with velocity, approaching infinity as an object approaches the speed of light. Forget whatever you heard; that formulation of special relativity is archaic and ugly. The mass m of an object as far as we are concerned is its *rest mass,* or the mass we would measure if we were at rest with respect to the object.

Rest mass is a scalar in that although different observers who are all moving at different speeds with respect to the object may, depending on the nature of their measuring apparati, measure different masses for an object, they all can agree on what its mass would be if they were at rest with respect to it. In this respect rest mass is like the proper time scalar: the only observers whose clocks actually measure the proper time between two events are the observers for whom the two events happen in the same place. But all observers agree on what that proper time is.

The 4-momentum $\vec{p}$ is thus

$$\begin{aligned}\vec{p} &\equiv& m\,\vec{u} \\ &=& (\gamma\,m\,c, \gamma\,m\,v_x, \gamma\,m\,v_y, \gamma\,m\,v_z) \\ &=& (\gamma\,m\,c, \gamma\,m\,\boldsymbol{v})\end{aligned} \tag{6.10}$$

Again, by switching into the rest frame of the particle, we find that $|\vec{p}| = m\,c$. This is also obvious because $\vec{p} = m\,\vec{u}$ and $|\vec{u}| = c$. As with 4-velocity, it is strange but true that the magnitude of the 4-momentum does not depend on speed. But of course it cannot, because speeds are relative.

Why introduce all these 4-vectors, and in particular the 4-momentum? In non-relativistic mechanics, 3-momentum is conserved. However, by Einstein's principle, all the laws of physics must be true in all uniformly moving reference frames. Because only scalars and 4-vectors are truly frame-independent, relativistically invariant conservation of momentum must take a slightly different form: in all interactions, collisions and decays of objects, *the total 4-momentum is conserved*. Furthermore, its time component is energy E/c (we must divide by c to give it the same dimensions as momentum) and its spatial components make up a correct, relativistic expression for the 3-momentum $\boldsymbol{p}$. We are actually re-defining E and $\boldsymbol{p}$ to be

$$\begin{aligned}E &\equiv& \gamma\,m\,c^2 \\ \boldsymbol{p} &\equiv& \gamma\,m\,\boldsymbol{v}\end{aligned} \tag{6.11}$$

Please forget any other expressions you learned for E or $\boldsymbol{p}$ in non-relativistic mechanics. Those other expressions are only good when speeds are much smaller than the speed of light.

A very useful equation suggested by the new, correct expressions for E and $\boldsymbol{p}$ is

$$\boldsymbol{v} = \frac{\boldsymbol{p}\,c^2}{E} \tag{6.12}$$

By taking the magnitude-squared of $\vec{p}$ we get a relation between m, E and $p \equiv |\boldsymbol{p}|$,

$$|\vec{p}|^2 = m^2\,c^2 = \left(\frac{E}{c}\right)^2 - p^2 \tag{6.13}$$

which, after multiplication by c^2 and rearrangement becomes

$$E^2 = m^2\,c^4 + p^2\,c^2 \tag{6.14}$$

This is the famous equation of Einstein's, which becomes $E = m\,c^2$ when the particle is at rest ($p = 0$).[‡]

If we take the low-speed limits, we should be able to reconstruct the non-relativistic expressions for energy E and momentum $\boldsymbol{p}$. In the low-speed limit $\beta \equiv v/c \ll 1$,

[‡]A friend of mine once was passed by a youth-filled automobile, the contents of which identified him as a physicist and shouted "Hey nerd: $E = mc^2$!" What has just been discussed explains why he ran down the street after the automobile shouting "Only in the rest frame!"

and we will make use of the fact that for small ϵ, $(1+\epsilon)^n \approx 1 + n\,\epsilon$. At low speed,

$$\begin{aligned}\boldsymbol{p} &=& m\,\boldsymbol{v}\,(1 - \beta^2)^{-1/2} \\ &\approx& m\,v + \frac{1}{2}\,m\,\frac{v^2}{c^2}\,\boldsymbol{v} \\ &\approx& m\,\boldsymbol{v} \\ E &=& m\,c^2\,(1 - \beta^2)^{-1/2} \\ &\approx& m\,c^2 + \frac{1}{2}\,m\,v^2\end{aligned} \tag{6.15}$$

i.e., the momentum has the classical form, and the energy is just Einstein's famous $m\,c^2$ plus the classical kinetic energy $m\,v^2/2$. But remember, these formulae only apply when $v \ll c$.

Conservation of 4-momentum is just like conservation of 3-momentum in non-relativistic mechanics. All the 4-momenta of all the components of the whole system under study are summed before the interaction, and they are summed afterwards. No matter what the interaction, as long as the whole system has been taken into account (i.e. the system is *isolated*), the total 4-momentum $\vec{p}$ before must equal the total 4-momentum $\vec{q}$ after. In effect this single conservation law $\vec{p} = \vec{q}$ summarizes four individual conservation laws, one for each component of the 4-momentum.

6.5   Collisions

It is now time to put conservation of 4-momentum into use by solving some physics problems. The essential technique is to sum up the total 4-momentum before and total 4-momentum after and set them equal. But just as in non-relativistic mechanics, there are tricks to learn and there are easy and difficult ways of approaching each problem.

In non-relativistic mechanics, collisions divide into two classes: elastic and inelastic. In elastic collisions, both energy and 3-momentum are conserved. In inelastic collisions, only 3-momentum is conserved. Energy is not conserved because some of the initial kinetic energy of the bodies or particles gets lost to heat or internal degrees of freedom. In relativistic mechanics, 4-momentum, and in particular the time component or energy, is conserved in *all* collisions; no distinction is made between elastic and inelastic collisions. As we will see, this is because the *correct,* relativistic expression we now use for energy takes all these contributions into account.

In Figure 6.1, a ball of putty of mass m is travelling at speed v towards another ball of putty, also of mass m, which is at rest. They collide and stick forming a new object with mass M' travelling at speed v'. In a non-relativistic world, M' would be $2m$ and v' would be $v/2$, a solution that conserves non-relativistic momentum but not non-relativistic energy; classically this collision is inelastic. But in a relativistic world we find that the non-relativistic predictions for v' and M' are not correct and both energy and 3-momentum will be conserved.
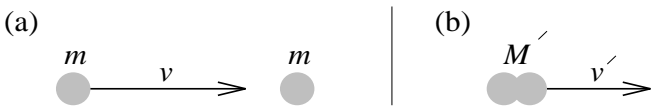
(a)

(b)

Figure 6.1: (a) A ball of putty of mass m travels at speed v towards an identical ball which is at rest. (b) After the collision the balls are stuck together and the combined lump has mass M' and speed v'.

Before the collision, the 4-momentum of the moving ball is $\vec{p}_m = (\gamma\, m\, c, \gamma\, m\, v, 0, 0)$, where I have aligned the x-axis with the direction of motion, and of course $\gamma \equiv (1-v^2/c^2)^{-1/2}$. The 4-momentum of the stationary ball is $\vec{p}_s = (m\, c, 0, 0, 0)$, so the total 4-momentum of the system is

$$\vec{p} = \vec{p}_m + \vec{p}_s = ([\gamma + 1]\, m\, c, \gamma\, m\, v, 0, 0) \qquad (6.16)$$

After the collision, the total 4-momentum is simply

$$\vec{q} = (\gamma'\, M'\, c, \gamma'\, M'\, v', 0, 0) \qquad (6.17)$$

where $\gamma' \equiv (1 - v'^2/c^2)^{-1/2}$.

By conservation of 4-momentum, $\vec{q} = \vec{p}$, which means that the two 4-vectors are equal, component by component, or

$$\begin{aligned} \gamma'\, M'\, c &= [\gamma + 1]\, m\, c \\ \gamma'\, M'\, v' &= \gamma\, m\, v \end{aligned} \qquad (6.18)$$

The ratio of these two components should provide v'/c; we find

$$v' = \frac{\gamma\, v}{\gamma + 1} > \frac{v}{2} \qquad (6.19)$$

The magnitude of $\vec{q}$ should be $M'\, c$; we find

$$\begin{aligned} M'^2 &= [\gamma + 1]^2\, m^2 - \gamma^2\, m^2\, \frac{v^2}{c^2} \\ &= \left[1 + 2\,\gamma + \gamma^2 \left(1 - \frac{v^2}{c^2} \right) \right] m^2 \\ &= 2\,(\gamma + 1)\, m^2 \\ M' &= \sqrt{2\,(\gamma + 1)}\, m \\ &> 2\, m \end{aligned} \qquad (6.20)$$

So the non-relativistic answers are incorrect, and most disturbingly, the mass M' of the final product is greater than the sum of the masses of its progenitors, $2\, m$.

Where does the extra rest mass come from? The answer is energy. The collision is classically inelastic. This means that some of the kinetic energy is lost. But energy is conserved, so the energy is not actually lost, it is just converted into other forms, like heat in the putty, or rotational energy of the combined clump of putty, or in vibrational waves or sound traveling through the putty. Strange as it may sound, this internal energy actually *increases the mass* of the product of the collision.

The consequences of this are strange. For example, a brick becomes more massive when one heats it up. Or,

a tourist becomes less massive as he or she burns calories climbing the steps of the Eiffel Tower.[§] Or, a spinning football hits a football player with more force than a non-spinning one. All these statements are true, but it is important to remember that the effect is very very small unless the internal energy of the object in question is on the same order as $m\, c^2$. For a brick of 1 kg, that energy is 10^{20} Joules, or 3×10^{13} kWh, or my household energy consumption over about ten billion years (roughly the age of the Universe). For this reason, macroscopic objects (like bricks or balls of putty) cannot possibly be put into states of relativistic motion in Earth-bound experiments. Only subatomic and atomic particles can be accelerated to relativistic speeds, and even these require huge machines (accelerators) with huge power supplies.

• **Problem 6–5:** *Suppose the two balls of putty in Figure 6.1 do not hit exactly head-on but rather at a slight perpendicular displacement, so in the final state the combined lump is spinning? How will this affect the final speed v'? And the final mass M'? Imagine now that you stop the combined lump from spinning—will its mass be greater than, equal to, or less than M'?*

6.6   Photons and Compton scattering

Can something have zero rest mass? If we blindly substitute $m = 0$ into Einstein's equation $E^2 = m^2\, c^4 + p^2\, c^2$ we find that $E = p\, c$ for a particle with zero rest mass (here p is the magnitude of the 3-momentum). But $v = p\, c^2/E$, so such massless particles would always have to travel at $v = c$, the speed of light. Strange.

Of course photons, or particles of light, have zero rest mass, and this is "why" they always travel at the speed of light. The magnitude of a photon's 4-momentum is zero, but this does not mean that the components are all zero; it just means that when the magnitude is calculated, the time component squared, E^2/c^2, is exactly cancelled out by the sum of the space components squared, $p_x^2 + p_y^2 + p_z^2 = |\boldsymbol{p}|^2$. Thus the photon may be massless, but it carries momentum and energy, and it should obey the law of conservation of 4-momentum. This was beautifully predicted and tested in the famous Compton scattering experiment. We outline the theory behind this experiment here.

Figure 6.2 shows the schematic for Compton scattering. A photon of initial 3-momentum magnitude Q (or energy Qc) approaches an electron of mass m that is essentially at rest. The photon scatters off of the electron, leaving at some angle θ to the original direction of motion, and with some new momentum Q' (or energy $Q'c$). The electron leaves at some other angle ϕ and some speed v. The idea of the experiment is to beam photons of known momentum Q at a target of stationary electrons, and measure the momenta Q' of the scattered photons as a function of scattering angle θ. We therefore want to derive an expression for Q' as a function of θ.

[§]Relativity does *not* provide the principal reason that one can lose weight by exercising; you do the math.
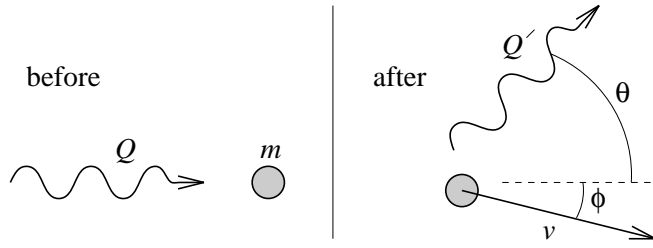
Figure 6.2: Before and after pictures for Compton scattering.

Before the collision the 4-momenta of the photon and electron are

$$\vec{p}_\gamma = (Q, Q, 0, 0) \tag{6.21}$$

$$\vec{p}_e = (m\,c, 0, 0, 0) \tag{6.22}$$

respectively, and after they are

$$\vec{q}_\gamma = (Q', Q'\cos\theta, Q'\sin\theta, 0) \tag{6.23}$$

$$\vec{q}_e = (\gamma\,m\,c, \gamma\,m\,v\cos\phi, -\gamma\,m\,v\sin\phi, 0) \tag{6.24}$$

respectively, where we have aligned coordinates so the initial direction of the photon is the x-direction, and the scatter is in the x-y plane. The conservation law is

$$\vec{p}_\gamma + \vec{p}_e = \vec{q}_\gamma + \vec{q}_e \tag{6.25}$$

but there is a trick. We can move both the photon 4-momenta to one side and both the electron momenta to the other and square (where $\vec{a}^2$ is just $\vec{a}\cdot\vec{a}$):

$$(\vec{p}_\gamma - \vec{q}_\gamma)^2 = (\vec{q}_e - \vec{p}_e)^2 \tag{6.26}$$

$$\vec{p}_\gamma\cdot\vec{p}_\gamma + \vec{q}_\gamma\cdot\vec{q}_\gamma - 2\,\vec{p}_\gamma\cdot\vec{q}_\gamma = \vec{p}_e\cdot\vec{p}_e + \vec{q}_e\cdot\vec{q}_e - 2\,\vec{p}_e\cdot\vec{q}_e \tag{6.27}$$

For all photons $\vec{p}\cdot\vec{p} = 0$ and for all electrons $\vec{p}\cdot\vec{p} = m^2\,c^2$. Also, in this case, $\vec{p}_\gamma\cdot\vec{q}_\gamma = Q\,Q' - Q\,Q'\cos\theta$ and $\vec{p}_e\cdot\vec{q}_e = \gamma\,m^2\,c^2$, so

$$-2\,Q\,Q'\,(1 - \cos\theta) = 2\,(1 - \gamma)\,m^2\,c^2 \tag{6.28}$$

But by conservation of energy, $(\gamma - 1)\,m\,c$ is just $Q - Q'$, and $(a - b)/ab$ is just $1/b - 1/a$, so we have what we are looking for:

$$\frac{1}{Q'} - \frac{1}{Q} = \frac{1}{m\,c}\,(1 - \cos\theta) \tag{6.29}$$

This prediction of special relativity was confirmed in a beautiful experiment by Compton (1923) and has been reconfirmed many times since by undergraduates in physics lab courses. In addition to providing quantitative confirmation of relativistic mechanics, this experimental result is a beautiful demonstration of the fact that photons, though massless, carry momentum and energy.

Quantum mechanics tells us that the energy E of a photon is related to its frequency ν by $E = h\,\nu$, and we know that for waves travelling at speed c, the frequency ν and wavelength λ are related by $\lambda = c/\nu$, so we can rewrite the Compton scattering equation in its traditional form:

$$\lambda' - \lambda = \frac{h}{m\,c}\,(1 - \cos\theta) \tag{6.30}$$

6.7   Mass transport by photons

Consider a box of length L and mass m at rest on a frictionless table. If a photon of energy $E \ll m\,c^2$ is emitted from one end of the box (as shown in Figure 6.3) and is absorbed by the other, what is the reaction of the box?
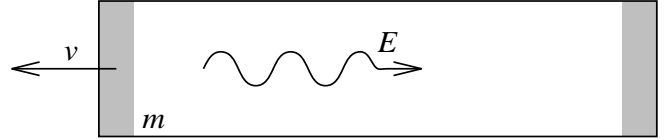
Figure 6.3: A thought experiment to demonstrate that there is a mass $\mu = E/c^2$ associated with a photon of energy E.

We know the previous section that a photon of energy E carries momentum E/c, so to conserve momentum, the emission of the photon must cause the box to slide backwards at a speed v given by $m\,v = -E/c$ (where it is okay to use the classical formula $m\,v$ for momentum because we stipulated $E \ll m\,c^2$ so $\gamma \ll 1$). The photon is absorbed a time Δt later, and the box must stop moving (again to conserve momentum). In time Δt, the box moves a distance

$$\Delta x_{\mathrm{b}} = v\,\Delta t = -\frac{E}{m\,c}\,\Delta t \tag{6.31}$$

and then stops, while the photon moves a distance

$$\Delta x_{\mathrm{p}} = c\,\Delta t = L - \frac{E}{m\,c}\,\Delta t \tag{6.32}$$

and then gets absorbed. Because the forces associated with the emission and absorption of the photon are totally internal to be box, we do not expect them to be able to transport the center of mass of the box (see, e.g., Frautschi et al., 1986, Chapter 11 for a non-relativistic discussion of this—it is a consequence of conservation of momentum). But because the box moved, the center of mass can only have remained at rest if the photon transported some mass μ from one end of the box to the other. To preserve the center of mass, the ratio of masses, μ/m must be equal to the ratio of their displacements $\Delta x_{\mathrm{b}}/\Delta x_{\mathrm{p}}$, so

$$\mu = m\,\frac{\Delta x_{\mathrm{b}}}{\Delta x_{\mathrm{p}}} = \frac{E}{c^2} \tag{6.33}$$

The transmission of the photon thus transports a mass $\mu = E/c^2$.

This does not mean that the photon is massive. The *rest mass* of a photon is zero. It only shows that when a photon of energy E is emitted, the emitter loses mass $\Delta m = E/c^2$ and when it is absorbed the absorber gains mass $\Delta m = E/c^2$.

● **Problem 6–6:** *In Chapter 5 we learned that no signal can travel through a solid body at a speed faster than that of light. The part of the box which absorbs the photon, therefore, won't know that a photon has been emitted*

from the other end until the photon actually arrives[¶]! Recast this argument for mass transport by photons into a form which does not rely on having a box at all.

6.8   Particle production and decay

• **Problem 6–7:** *A particle of mass M, at rest, decays into two smaller particles of masses m_1 and m_2. What are their energies and momenta?*

Before decay, the 4-momentum is $(E/c, \boldsymbol{p}) = (Mc, \boldsymbol{0})$. After, the two particles must have equal and opposite 3-momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in order to conserve 3-momentum. Define $p \equiv |\boldsymbol{p}_1| = |\boldsymbol{p}_2|$; in order to conserve energy $E_1 + E_2 = E = M c^2$ or

$$\sqrt{p^2 + m_1^2 c^2} + \sqrt{p^2 + m_2^2 c^2} = M c \qquad (6.34)$$

This equation can be solved (perhaps numerically—it is a quartic) for p and then $E_1 = \sqrt{m_1^2 c^4 + p^2 c^2}$ and $E_2 = \sqrt{m_2^2 c^4 + p^2 c^2}$.

• **Problem 6–8:** *Solve the above problem again for the case $m_2 = 0$. Solve the equations for p and E_1 and then take the limit $m_1 \to 0$.*

• **Problem 6–9:** *If a massive particle decays into photons, explain using 4-momenta why it cannot decay into a single photon, but must decay into two or more. Does your explanation still hold if the particle is moving at high speed when it decays?*

• **Problem 6–10:** *A particle of rest mass M, travelling at speed v in the x-direction, decays into two photons, moving in the positive and negative x-direction relative to the original particle. What are their energies? What are the photon energies and directions if the photons are emitted in the positive and negative y-direction relative to the original particle (i.e., perpendicular to the direction of motion, in the particle's rest frame).*

6.9   Velocity addition (revisited) and the Doppler shift

The fact that the 4-momentum transforms according to the Lorentz transformation makes it very useful for deriving the velocity addition law we found in Section 4.4. In frame $\mathcal{S}$, a particle of mass m moves in the x-direction at speed v_1, so its 4-momentum is

$$\vec{p} = (\gamma_1 \, m \, c, \gamma_1 \, m \, v_1, 0, 0) \qquad (6.35)$$

where $\gamma_1 \equiv (1 - v_1^2/c^2)^{-1/2}$. Now switch to a new frame $\mathcal{S}'$ moving at speed $-v_2$ in the x-direction. In this frame the 4-momentum is

$$\begin{aligned} \vec{p'} &= \Big(\gamma_2 \, \gamma_1 \, m \, c + \frac{v_2}{c} \, \gamma_2 \, \gamma_1 \, m \, v_1, \\ &\qquad \gamma_2 \, \gamma_1 \, m \, v_1 + \frac{v_2}{c} \, \gamma_2 \, \gamma_1 \, m \, c, 0, 0\Big) \end{aligned} \qquad (6.36)$$

[¶]I acknowledge French (1966) for pointing out this problem with the above argument.

The speed is just the ratio of x and t-components, so

$$\begin{aligned} \frac{v'}{c} &= \frac{\gamma_2 \, \gamma_1 \, m \, v_1 + \gamma_2 \, \gamma_1 \, m \, v_2}{\gamma_2 \, \gamma_1 \, m \, c + \gamma_2 \, \gamma_1 \, m \, v_1 \, v_2/c} \\ v' &= \frac{v_1 + v_2}{1 + v_1 \, v_2/c^2} \end{aligned} \qquad (6.37)$$

This is a much simpler derivation than that found in Section 4.4!

Consider now a photon in $\mathcal{S}$ with 4-momentum $\vec{q} = (Q, Q, 0, 0)$. In frame $\mathcal{S}'$ the 4-momentum is

$$\vec{q'} = \left(\gamma_2 \, Q + \frac{v_2}{c} \, \gamma_2 \, Q, \gamma_2 \, Q + \frac{v_2}{c} \, \gamma_2 \, Q, 0, 0\right) \qquad (6.38)$$

Clearly this is still travelling at the speed of light (as it must) but now its new 3-momentum is

$$Q' = \gamma_2 \left(1 + \frac{v_2}{c}\right) Q = \sqrt{\frac{1 + v_2}{1 - v_2}} \, Q \qquad (6.39)$$

This change in momentum under a boost is the *Doppler shift*, and is discussed in more detail in the next Chapter.

6.10   4-force

We now have 4-velocity and 4-momentum, and we know how to use them. If we want to construct a complete, invariant dynamics, analogous to Newton's laws but valid in all reference frames, we are going to need 4-acceleration and 4-force. Recall that we defined a 4-vector to be a four component object that transforms according to the Lorentz transformation. For this reason, the 4-velocity $\vec{u}$ and 4-momentum $\vec{p}$ are defined in terms of derivatives with respect to proper time τ rather than coordinate time t. The definitions are $\vec{u} \equiv d\vec{x}/d\tau$ and $\vec{p} \equiv m\vec{u}$, where $\vec{x}$ is spacetime position and m is rest mass.

For this same reason, if we want to define a 4-vector form of acceleration, the 4-acceleration $\vec{a}$, or a 4-vector force, 4-force $\vec{K}$, we will need to use

$$\begin{aligned} \vec{a} &\equiv \frac{d\vec{u}}{d\tau} \qquad (6.40) \\ \vec{K} &\equiv \frac{d\vec{p}}{d\tau} \qquad (6.41) \end{aligned}$$

Because $\vec{p} = (E, \boldsymbol{p})$, we have

$$\vec{K} = \left(\frac{dE}{d\tau}, \frac{d\boldsymbol{p}}{d\tau}\right) . \qquad (6.42)$$

Because $\Delta t = \gamma \, \Delta \tau$ (where, as usual, $\gamma \equiv (1 - v^2/c^2)^{-1/2}$), the spatial part of the 4-force is related to Newton's force $\boldsymbol{F}$, defined as $\boldsymbol{F} \equiv d\boldsymbol{p}/dt$, by

$$\frac{d\boldsymbol{p}}{d\tau} = \gamma \, \boldsymbol{F} \qquad (6.43)$$

Also, if the rest mass m of the object in question is a constant (not true if the object in question is doing work,

because then it must be using up some of its rest energy!),
we have that

$$
\begin{aligned}
\vec{p} \cdot \vec{p} &= m^2 c^2 \\
\frac{d}{d\tau}(\vec{p} \cdot \vec{p}) &= 0 \\
\frac{d\vec{p}}{d\tau} \cdot \vec{p} + \vec{p} \cdot \frac{d\vec{p}}{d\tau} &= 0 \\
\vec{p} \cdot \vec{K} &= 0
\end{aligned}
\tag{6.44}
$$

i.e., if the rest mass is not changing then $\vec{p}$ and $\vec{K}$ are
orthogonal. In 3+1-dimensional spacetime, orthogonality
is something quite different from orthogonality in 3-space:
it has nothing to do with 90° angles.

The 4-force is only brought up here to whet the
reader's appetite. We will actually have to make use of it
in the (currently non-existent) Chapter on electricity.

Chapter 7

Optics and apparent effects: special relativity applied to astronomy

Up to now, we have always stipulated that observers making measurements are endowed with divine knowledge and excellent data analysis skills (recall Section 2.2). For example, in Chapter 2, when E measured the rate of D's clock, she did not simply measure the time between light pulses she received, she corrected them for their light-travel times in getting from D's clock to her eyes. The corrections E made to the arrival times were only possible because E was informed of D's trajectory before the experiment. Unfortunately, in many experiments, we do not know in advance the trajectories of the objects we are studying. This is especially true in astronomy, a subject which, among other things, attempts to reconstruct a 3+1-dimensional history of the Universe from a set of 2-dimensional telescope pictures which span a very brief duration in time (in comparison with the age of the Galaxy or Universe).

In this chapter we discuss the appearance of objects to real observers.

7.1 Doppler shift (revisited)

Consider an object moving with respect to the Earth and which we are observing from Earth. Without loss of generality, we can choose the coordinate system for the Earth's rest frame that puts the Earth at the spatial origin, the moving object a distance D away on the positive x-axis, and puts the object's trajectory in the x-y plane. Its velocity vector $\boldsymbol{v}$ makes an angle θ with the line of sight, as shown in Figure 7.1.
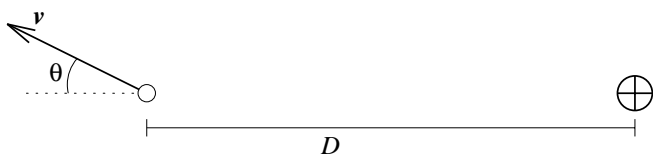
Figure 7.1: An object moving at relativistic velocity $\boldsymbol{v}$ with respect to the Earth (symbolized by "$\oplus$") at an angle θ to the line of sight. Note that this is a diagram of space rather than spacetime.

Let us imagine that the moving object emits pulses of light at intervals of proper time $\Delta\tau_e$, and that its distance from us D is much greater than $c\,\Delta\tau_e$. The question we want to answer is this: If the light pulse emitted at time 0 arrives at Earth at time $t = D/c$, how much later does the next pulse arrive?

The next pulse is emitted a time $\Delta t_e = \gamma\,\Delta\tau_e$, later (where $\gamma \equiv (1 - \beta^2)^{-1/2}$, $\beta \equiv v/c$, and $v \equiv |\boldsymbol{v}|$), at which time the object is $\Delta x = v\,\Delta t_e\,\cos\theta$ further away, so the flash takes additional time $\Delta x/c$ to get to us. The time interval Δt_r between reception of the flashes is therefore

$$
\begin{aligned}
\Delta t_r &= \Delta t_e + \frac{v}{c}\,\Delta t_e\,\cos\theta \\
&= (1 + \beta\cos\theta)\,\gamma\,\Delta\tau_e \ . \qquad (7.1)
\end{aligned}
$$

If the motion is basically away from the Earth ($\theta < \pi$), the time interval Δt_r is longer than $\Delta\tau_e$. The analysis still holds if we take the two events not to be flashes, but successive crests of an electromagnetic wave coming from the object. The observed period is longer than the rest-frame period; the observed frequency is lower than the rest-frame frequency; the light is shifted to the red.

It is customary in astronomy to define a dimensionless *redshift* z by

$$
\begin{aligned}
(1 + z) &\equiv \frac{\Delta t_r}{\Delta\tau_e} \\
&= \gamma\,(1 + \beta\cos\theta) \ . \qquad (7.2)
\end{aligned}
$$

In the simple case $\theta = 0$ (radial motion) the redshift is given by

$$
(1 + z) = \gamma\,(1 + \beta) = \sqrt{\frac{1 + \beta}{1 - \beta}} \ , \qquad (7.3)
$$

and when $\theta = \pi$ (inward radial motion) the redshift z is *negative,* we call it a *blueshift* and it is given by

$$
(1 + z) = \gamma\,(1 - \beta) = \sqrt{\frac{1 - \beta}{1 + \beta}} \ . \qquad (7.4)
$$

Even when the motion is perfectly tangential, $\theta = \pi/2$, there is a redshift which originates solely in the γ factor. This is known as the *second-order* redshift and

it has been observed in extremely precise timing of high-velocity pulsars in the Galaxy. Of course all of these redshift effects are observed and have to be corrected-for in tracking and communication between artificial satellites.

Interestingly, the Doppler shift computed here, for the ratio of time intervals between photon arrivals in two different frames, is just the reciprocal of the Doppler shift formula computed in Section 6.9, for the ratio of photon energies in two different frames. In quantum mechanics, the energy of a photon is proportional to the frequency of light, which is the reciprocal of the time interval between arrivals of successive wave crests. Quantum mechanics and special relativity would be inconsistent if we did not find the same formula for these two ratios. Does this mean that special relativity requires that a photon's energy be proportional to its frequency?

• **Problem 7–1:** *The* [O II] *emission line with rest-frame wavelength* $\lambda_0 = 3727 \,Å$ *is observed in a distant galaxy to be at* $\lambda = 9500 \,Å$. *What is the redshift* z *and recession speed* β *of the galaxy?*

Light travels at speed c, so the observed wavelength λ is related to the observed period T by $cT = \lambda$. The rest-frame wavelength λ_0 is related to the rest frame period τ by $c\tau = \lambda_0$. So

$$(1+z) \equiv \frac{T}{\tau} = \frac{\lambda}{\lambda_0} = \frac{9500\,Å}{3727\,Å}\;; \qquad (7.5)$$

$z = 1.55$. Assuming the velocity is radial,

$$
\begin{aligned}
(1+z) &= \sqrt{\frac{1+\beta}{1-\beta}} \\
(1+z)^2 - \beta\,(1+z)^2 &= 1+\beta \\
\beta &= \frac{(1+z)^2 - 1}{(1+z)^2 + 1}\,, \qquad (7.6)
\end{aligned}
$$

in this case we get $\beta = 0.73$. The galaxy is receding from us at $0.73c$.

7.2 Stellar Aberration

Imagine two observers, Ursula (U) and Virginia (V), both at the same place, observing the same star, at the same time, but with V moving in the x-direction at speed v relative to U. In U's frame, the star is a distance r away and at an elevation angle θ with respect to the x-axis. Light travels at speed c, so for any photon coming from the star, the 4-displacement $\Delta\vec{x}$ between the event of emission E and observation O in U's frame is

$$
\begin{aligned}
\Delta\vec{x} &= (c\Delta t, \Delta x, \Delta y, \Delta z) \\
&= (-r, r\cos\theta, r\sin\theta, 0) \qquad (7.7)
\end{aligned}
$$

where the time component is negative because emission happens before observation. We apply the Lorentz transformation to get the components in V's frame

$$
\begin{aligned}
\Delta\vec{x} &= (c\Delta t', \Delta x', \Delta y', \Delta z') \\
&= (-\gamma\,r - \gamma\,\beta\,r\cos\theta, \gamma\,r\cos\theta + \gamma\,\beta\,r, r\sin\theta, 0) \\
&\qquad\qquad (7.8)
\end{aligned}
$$

where, as usual, $\beta \equiv v/c$ and $\gamma \equiv (1 - v^2/c^2)^{-1/2}$. Since the photons also travel at speed c in V's frame, we can re-write this in terms of the distance r' to the star and elevation angle θ' in V's frame:

$$\Delta\vec{x} = (-r', r'\cos\theta', r'\sin\theta', 0) \qquad (7.9)$$

Solving for θ',

$$\cos\theta' = \frac{\cos\theta + \beta}{1 + \beta\cos\theta} \qquad (7.10)$$

i.e., V observes the star to be at a different angular position than that at which U does, and the new position does not depend on the distance to the star.

This effect is *stellar aberration* and it causes the positions on the sky of celestial bodies to change as the Earth orbits the Sun.[*] The Earth's orbital velocity is $\sim 30\ \mathrm{km\,s^{-1}}$ ($\beta = 10^{-4}$), so the displacement of an object along a line of sight perpendicular to the plane of the orbit (i.e., $\cos\theta = 0$) is on the order of 10^{-4} radians or ~ 20 arcseconds, a small angle even in today's telescopes. Despite this, the effect was first observed in a beautiful experiment by Bradley in 1729.[†]

Notice that as the speed v is increased, the stars are displaced further and further towards the direction of motion. If U is inside a uniform cloud of stars and at rest with respect to them, V will see a non-uniform distribution, with a higher density of stars in the direction of her motion relative to the star cloud and a lower density in the opposite direction.

7.3 Superluminal motion

It is observed that two components of the radio galaxy 3C 273 are moving apart at $\mu = 0.8$ milliarcseconds per year (Pearson et al 1981; recall that a milliarcsecond is $1/1000$ of $1/3600$ of a degree). From the known rate of expansion of the Universe and the redshift of the radio galaxy, its distance[‡] D from the Milky Way (our own galaxy) has been determined to be 2.6×10^9 light years (a light year is the distance light travels in one year). If we multiply μ by D we get the tangential component of the relative velocity of the two components. Because there can also be a radial component, the velocity component we derive will be a lower limit on the speed of the object. Converting to radians we find $\mu = 4 \times 10^{-9}$ radians per year, so the tangential component of the velocity is roughly 10 light years per year! This is faster than twice the speed of light, the maximum relative speed at which we should ever observe two objects to move. Relative speeds exceeding $2c$ have now been observed in many radio galaxies, and recently even in a jet of material flowing out of a star in our own galaxy (Hjellming & Rupen

---

[*]Do not confuse this effect with parallax, which also causes the positions to change, but in a manner which depends on distance.

[†]The paper is Bradley (1729); an excellent description and history of the experiment is Shankland (1964).

[‡]In cosmology, there are many different ways of defining the "distance" betweeen two objects, reviewed by Weinberg (1972, Chapter 14). The "proper motion distance" is used in this context.

1995); the effect has been dubbed *superluminal motion.* Is relativity wrong and can things really exceed the speed of light?

Figure 7.2 depicts an object moving at a relativistic speed $v = |\boldsymbol{v}|$ at an angle θ to the line of sight. The object is nearly moving directly towards the Earth, so θ is close to π radians or $180°$. The object emits flashes at events A and B, which are separated in time by Δt_e in the Earth's rest frame. The distance between the events is much smaller than the distance D of the object from the Earth.
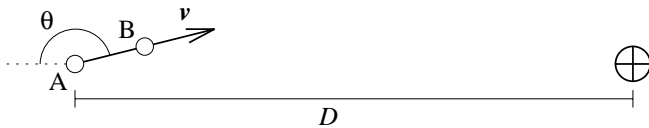
Figure 7.2: An object moving at relativistic velocity $\boldsymbol{v}$ on a trajectory that is nearly straight towards the Earth. The object emits flashes at points A and B.

What is the time interval Δt_r between the receptions of the two flashes at the Earth? Flash A takes time D/c to get to us, but flash B takes only $D/c + (v\,\Delta t_e \cos\theta)/c$ to get to us because the object is closer (note that $\cos\theta$ is negative). So

$$\Delta t_r = \Delta t_e + \beta\,\Delta t_e \cos\theta \;, \qquad (7.11)$$

where $\beta \equiv v/c$. The tangential separation of events A and B as seen from the Earth is $\Delta y = v\,\Delta t_e \sin\theta$, so the inferred tangential velocity component is

$$v_{\text{inferred}} = \frac{\Delta y}{\Delta t_r} = \frac{\beta \sin\theta}{1 + \beta \cos\theta}\, c \;, \qquad (7.12)$$

which can be much bigger than c if $\beta \approx 1$ and $\cos\theta \approx -1$.

(It is worthy of note that there are many other possible explanations for observed superluminal motions. If the radio galaxy contains a huge "searchlight" that sweeps its beam across intergalactic material, the speed of the patch of illumination can certainly exceed the speed of light. Galaxies can act as gravitational "lenses" which distort and magnify background objects; this magnification can make slowly-moving objects appear superluminal. The moving patches could be foreground objects, although this now appears *very* unlikely.)

• **Problem 7–2:** *What is the minimum possible value of β that could account for the observed proper motion in 3C 273? Assume that one component is not moving tangentially with respect to the Earth and the other is.*

7.4   Relativistic beaming

Consider an object emitting photons in all directions isotropically. The *brightness* of the object is proportional to the amount of radiation (energy per unit time) which the object emits into the pupil of the observer's eye or telescope, and inversely proportional to the solid angle (angular area, measured in square arcseconds, square degrees, or steradians) occupied by the object. The dimensions of brightness are energy per unit time per unit solid angle. Thus if two objects emit the same amount of light, the more compact one is brighter. Brightness is a useful quantity in astronomy because it is independent of distance: as a lightbulb is moved away from an observer, the amount of light from the bulb entering the observer's eye or telescope goes down as the inverse square of the distance, but the solid angular size of the bulb also goes down as the inverse square of the distance. The brightness is constant.

Okay, the brightness of an object is independent of distance, but how does it depend on how the object is *moving* relative to the observer? Doppler shift (Sections 6.9 and 7.1) affects both the energy E (or momentum Q) of the photons and the rate of production Γ of the photons (i.e., number of photons emitted per unit time). In addition, the photon *directions* are different for the observer than for someone in the rest frame of the object (as in stellar aberration, Section 7.2), so the fraction of emitted photons entering the observer's eye or telescope will also be affected by the object's speed and direction. For the same reason that in stellar aberration (Section 7.2) observed star positions are shifted into the direction of motion of the observer, emitted photons are "beamed" into the direction of motion of the emitter.

Say the emitting object is at rest in frame $\mathcal{S}'$, the rest frame, but moving at speed $v = \beta c$ in the positive x-direction in frame $\mathcal{S}$, the frame of the observer. In its rest frame, the object emits photons of energy $E' = Q'c$ at rate Γ' (photons per unit time). A photon emitted in a direction θ' relative to the x-axis in frame $\mathcal{S}'$ has 4-momentum

$$\vec{p} = (Q', Q' \cos\theta', Q' \sin\theta', 0) \qquad (7.13)$$

where the y-direction has been chosen to make $p_z = 0$ (Section 6.6). In frame $\mathcal{S}$ it will have some different momentum Q and angle θ and the 4-momentum will be

$$\vec{p} = (Q, Q \cos\theta, Q \sin\theta, 0) \qquad (7.14)$$

but it must be related to the 4-momentum in $\mathcal{S}'$ by the Lorentz transformation, so

$$\begin{aligned} Q' &= \gamma Q - \gamma \beta Q \cos\theta \\ Q' \cos\theta' &= \gamma Q \cos\theta - \gamma \beta Q \end{aligned} \qquad (7.15)$$

The first equation is just the Doppler shift (Sections 4.4 and 7.1); the ratio gives

$$\cos\theta' = \frac{cos\theta - \beta}{1 - \beta \cos\theta} \qquad (7.16)$$

which is exactly the same as the stellar aberration equation (Section 7.2).

In the rest frame $\mathcal{S}'$ the object emits isotropically, so the rate per unit solid angle Ω (measured in steradians, or radians^2) is just

$$\frac{d\Gamma'}{d\Omega'} = \frac{\Gamma'}{4\pi} \qquad (7.17)$$

which is independent of θ. In the observer frame $\mathcal{S}$, however, this will no longer be true. Consider the solid-angular ring of angular width $d\theta$ at angle θ. This ring has solid angle

$$d\Omega = \sin\theta\, d\theta \qquad (7.18)$$

but the photons emitted into that ring in $\mathcal{S}$ are emitted into a different ring in $\mathcal{S}'$ with solid angle

$$d\Omega' = \sin\theta'\, d\theta' \qquad (7.19)$$

where θ and θ' are related by (7.16). Taking the derivative of (7.16)

$$
\begin{aligned}
\sin\theta'\, d\theta' &= \frac{\sin\theta\, d\theta}{1-\beta\,\cos\theta} + \frac{(\cos\theta-\beta)(\beta\,\sin\theta\, d\theta)}{(1-\beta\,\cos\theta)^2} \\
&= \sin\theta\, d\theta\left[\frac{1-\beta^2}{(1-\beta\,\cos\theta)^2}\right] \\
&= \frac{\sin\theta\, d\theta}{\gamma^2\,(1-\beta\,\cos\theta)^2} \qquad (7.20)
\end{aligned}
$$

so the ratio of solid angles is

$$\frac{d\Omega}{d\Omega'} = \gamma^2\,(1-\beta\,\cos\theta)^2 \qquad (7.21)$$

the square root of which is the ratio of energies E'/E (by the Doppler shift) or the ratio of rates of photon production Γ'/Γ (by the same). Putting it all together, since the inferred brightness is proportional to the energies times the rate divided by the solid angle, the ratio of brightness I/I' between the observer and rest frames is

$$\frac{I}{I'} = [\gamma\,(1-\beta\,\cos\theta)]^{-4} \qquad (7.22)$$

or in terms of redshift, $(1+z)^{-4}$!

• **Problem 7–3:** *Plot the observed brightness I as a function of angle θ according to an observer at rest in $\mathcal{S}$ observing an object radiating isotropically in its rest frame $\mathcal{S}'$.*

7.5   The appearance of passing objects

Consider a rectangular plank of rest dimensions[§] $X \times Y$ moving at speed $v = \beta c$ in the x-direction, perpendicular to the line of sight to a distant observer, as shown in Figure 7.3. The light coming from the corners marked A and B get to the observer before the light coming from corner C by a time interval Y/c. For this reason, at any instant of time, the plank will appear "rotated" to the observer, as you will show in the problems. There is a nice discussion of this apparent rotation effect in French (1966, pp. 149–152). The apparent rotation actually needs to be taken into account by astrophysicists modeling features in relativistic jets emitted by radio galaxies and stars (e.g., Lind & Blandford 1985).

_____

[§]The "rest dimensions" are the dimensions the object has in its rest frame.
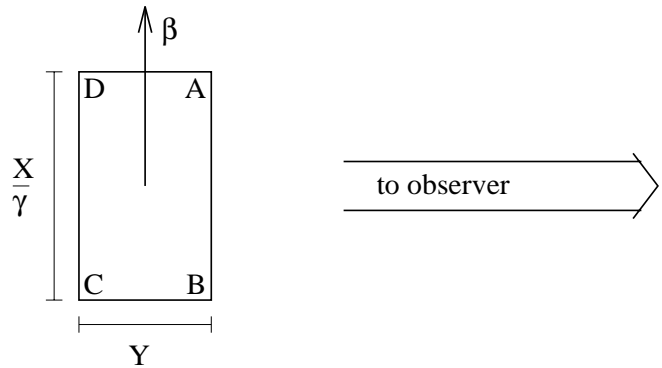
Figure 7.3: A plank of rest dimensions $X \times Y$ moves at speed β perpendicular to the line of sight to a distant observer.

• **Problem 7–4:** *What is the apparent position of corner C to the observer in Figure 7.3 at the time that the light from corners A and B reach the observer? From this information, as well as length contraction, compute the apparent locations of all four corners.*

• **Problem 7–5:** *Why doesn't the observer see corner D?*

7.6   A simpleminded cosmology

We know that the Universe is expanding. In fact, we know that except for a few, very close neighbours, other galaxies are receding from our own and recession speed is proportional to distance from us. This effect is known as the *Hubble flow,* named after the astronomer who first discovered it (Hubble, 1929). This Hubble flow is naturally explained by a simple cosmological scenario in which the Universe begins with an explosion, and this scenario does not require our galaxy to be at the center.

Consider an infinite Lorentz frame $\mathcal{S}'$ with a small rock at rest at the origin. At time $t' = 0$ in this frame, the rock explodes into countless tiny fragments with masses small enough to ensure that gravitational forces do not significantly affect the constant-velocity (speed and direction) trajectories. At time $t' > 0$, there is some distribution of fragments in space, with the faster-moving fragments further out from the explosion point. Because all fragment world lines are constant-velocity and pass through the event (0,0,0,0) in $\mathcal{S}'$, the vector displacement $\boldsymbol{r}'$ in frame $\mathcal{S}'$ of a fragment with velocity $\boldsymbol{v}'$ is given by $\boldsymbol{r}' = \boldsymbol{v}'\,t'$.

Now consider another frame $\mathcal{S}$ which also has the explosion at the origin, but which is moving along with one of the fragments not at rest in $\mathcal{S}'$. In $\mathcal{S}$, all the fragments have constant-velocity worldlines that pass through the event (0,0,0,0). Therefore in $\mathcal{S}$ also, the displacement $\boldsymbol{r}$ at time $t > 0$ of a fragment with velocity $\boldsymbol{v}$ is given by $\boldsymbol{r} = \boldsymbol{v}t$. That is, at any time $t > 0$, recession speed is proportional to distance from the origin even though the origin is not at (or even at rest with respect to) the center of the explosion.

If at time $t = t_0$ (now) we live on a fragment (the

Milky Way) ejected by a huge explosion (the Big Bang) which occurred at time $t = 0$, and the fragments are not heavy enough to have significantly affected each other's velocities via gravitational forces, then by the above argument we expect to see other nearby fragments (other galaxies) receding from us, with their recession speeds proportional to their distances from us; i.e. we expect a spherically symmetric Hubble flow even if we are not at the center of the Universe.

Of course when we look at an extremely distant object now, we are not seeing the object at its current position $\boldsymbol{r}(t_0)$, but rather at its position $\boldsymbol{r}(t_e)$ at time t_e when it emitted the light that is now reaching us. Also, we have no direct measure of the distance $r_e = |\boldsymbol{r}(t_e)|$, but we can infer it from the redshift z of the light that it emits in, say, its hydrogen recombination lines (the rest-frame frequencies of which we know). What is the relationship between z and r_e?
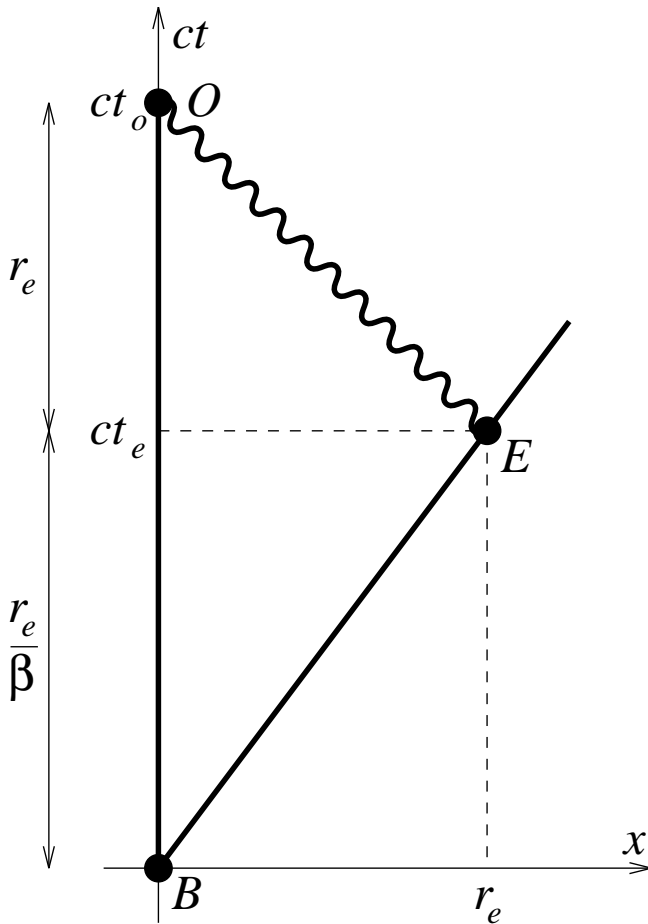
Figure 7.4: The spacetime diagram used to derive the redshift-distance relation in a simpleminded cosmology. World lines of the Earth (vertical) and the fragment (slope $1/\beta$) are shown. Event B is the big bang, E the emission of light and O its observation now on Earth.

Figure 7.4 is the spacetime diagram for a fragment moving in $\mathcal{S}$ (where $\mathcal{S}$ is our rest frame) at velocity $\boldsymbol{v}$, with coordinates aligned so that $\boldsymbol{v}$ points in the x-direction.

The Big Bang occurs at event B, the origin ($ct = x = 0$); the fragment emits light at event E ($ct = ct_e$, $x = r_e$); and we observe the light at event O, now ($ct = ct_0$, $x = 0$).

It should be obvious from the diagram that $ct_0 = r_e/\beta + r_e$, where $\beta = |\boldsymbol{v}|/c$ and that the proper time τ_{BE} elapsed for the fragment between B and E is given by $(c\tau_{BE})^2 = (r_e/\beta)^2 - r_e^2$. From these relations and the fact that the redshift z is given by $1 + z = t_0/\tau_{BE}$ (Section 7.1) it is easy to show that

$$r_e = ct_0 \frac{2z + z^2}{2(1+z)^2} \qquad (7.23)$$

(the student is encouraged to show this). It should be obvious, both from Figure 7.4 and the above equation, that the maximum value for r_e is $ct_0/2$ when $z \to \infty$, and that for small z, $r_e = ct_0 z$.

In addition to inference from redshift, the distance to a fragment can be determined several other ways. If one knows the size of the fragment, its angular diameter can be measured, and the ratio of the quantities should provide the distance r_e. For this reason, r_e is referred to as the *angular diameter distance*[¶] to the object, and is often denoted d_{A}. If the intrinsic luminosity L of a fragment is known, its flux F can be measured, and the relation $F = L/(4\pi r^2)$ can be used to determine a distance. However, the *luminosity distance* d_{L} determined in this way is different from d_{A} by four factors of $(1+z)$ because of the effect of redshift on brightness discussed in Section 7.4.

The cosmology presented in this section is a simple Milne cosmology, a more general version of which (including gravity) is described by Milne (1934). Most cosmologists now believe that the expansion of the Universe is governed by general relativity, but it is nonetheless true that most cosmological observations can be explained by this simple kinematic model.

[¶]Experienced cosmologists will notice that equation (7.23) is identical in form to the equation derived, via general relativity, for the angular diameter distance in a "spatially curved, isotropic, homogeneous, empty space." See, e.g., Weinberg (1972) or Peebles (1993) for the general-relativistic derivation.

References

Binney J. & Tremaine S., 1987. *Galactic Dynamics.* Princeton University Press, Princeton.

Bradley J., 1729. *Philosophical Transactions of the Royal Society* **35** 637.

Compton A. H., 1923. *Physical Review* **22** 409.

Einstein A., 1905. Zur Elektrodynamik bewegeter Körper. *Annalen der Physik* **17**. (English translation: On the electrodynamics of moving bodies, reprinted in Lorentz et al., 1923.)

Einstein A., 1911. Uber den Einfluss der Schwerkraft auf die Ausbreitung des Lichtes. *Annalen der Physik* **35**. (English translation: On the influence of gravitation on the propagation of light, reprinted in Lorentz et al., 1923.)

Frautschi S. C., Olenick R. P., Apostol T. M. & Goodstein D. L., 1986. *The Mechanical Universe.* Cambridge University Press, Cambridge.

French A. P., 1966. *Special Relativity.* W. W. Norton & Co., New York.

Hjellming R. M. & Rupen M. P., 1995, Episodic ejection of relativistic jets by the X-ray transient GRO J1655-40, *Nature* **375** 464–468.

Hubble E., 1929. A relation between distance and radial velocity among extra-galactic nebulae. *Proceedings of the National Acadamy of Sciences* **15** 168–173.

Jackson J. D., 1975. *Classical Electrodynamics,* 2 ed. John Wiley & Sons, New York.

Kleppner D. & Kolenkow R. J., 1973. *An Introduction to Mechanics.* McGraw–Hill, New York.

Lind K. R. & Blandford R. D., Semidynamical models of radio jets: Relativistic beaming and source counts, *Astrophysical Journal* **295** 358–367.

Lorentz H. A., 1904. Electromagnetic phenomena in a system moving with any velocity less than that of light. *Proceedings of the Academy of Sciences of Amsterdam* **6**. (Reprinted in Lorentz et al., 1923.)

Lorentz H. A., Einstein A., Minkowski H. & Weyl H., 1923. *The Principle of Relativity.* Perrett W. & Jeffery G. B., translators. Dover Publications Inc., New York.

Milne E. A., 1934. A Newtonian expanding universe. *Quarterly Journal of Mathematics* **5** 64–72.

Michelson A. A. & Morley E. W., 1887. On the relative motion of the Earth and the luminiferous ether. *American Journal of Science* **134** 333–345.

Newton I., 1730. *Opticks.* (Reprinted by Dover, New York, 1979.)

Particle Data Group, 1994. Review of particle properties. *Physical Review D* **50** 1173–1825.

Pearson T. J., Unwin S. C., Cohen M. H., Linfeld R. P., Readhead A. C. S., Seielstad G. A., Simon R. S. & Walker R. C., 1981, *Nature,* **290** 365–368.

Peebles P. J. E., 1993. *Principles of Physical Cosmology.* Princeton University Press, Princeton.

Poincaré H., 1900. Relations entre la physique expérimentale et la physique mathématique. *Rapports présentés au Congrès International de Physique de 1900* (Paris) **1** 1–29.

Purcell E. M., 1985. *Electricity and Magnetism.* McGraw-Hill, New York.

Rossi B. & Hall D. B., 1941. Variation of the rate of decay of mesotrons with momentum. *Physical Review* **59** 223–228.

Shankland R. S., 1964. Michelson-Morley Experiment. *American Journal of Physics* **32** 16–35.

Strang G., 1976, *Linear Algebra and its Applications.* Harcourt Brace Jovanovich, San Diego.

Weinberg S., 1972. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* John Wiley & Sons, New York.

Index