2. Special Relativity (SR) — Lorentz transformations

Lorentz transformations

- relate the coordinate systems of two inertial observers
- leave the "4-distance" invariant
- assuming linearity, they can be written as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu}$$

  – These are called inhomogeneous Lorentz transformations (Λ, a)

Homogeneous Lorentz transformations have $a^{\mu} = 0$

- They leave scalar products invariant: $(p'.q') = (p.q)$
- They describe 3 Rotations and 3 Boosts
  – compare with the Galilean transformations

2. Special Relativity (SR) — Lorentz transformations

Rotations are the same as in the Galilean transformations

For Boosts between O and O' let us align the coordinate systems:

- The origins of O and O' should be at the same place at $t = t' = 0$
- The constant relative velocity v between O and O' should point in the $\hat{x}$-direction for both, O and O'
- $\hat{y}$ ($\hat{z}$) should point in the same direction: $y' = y$ $(z' = z)$
- Only $ct = x^0$ and $x = x^1$ are affected by such a boost:
  $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}$ for either μ or ν being 2 or 3
- So with $p' = \Lambda p$ and $q' = \Lambda q$ we have $(p'.q') - (p.q) = 0$
- Since $y' = y$ and $z' = z$ we can ignore $\hat{y}$ and $\hat{z}$ in the equation

$$0 = (p'.q') - (p.q) = (p'^0 q'^0 - p'^1 q'^1) - (p^0 q^0 - p^1 q^1)$$

Determining Boosts

$$
\begin{aligned}
0 &= (\Lambda^0_{\ 0} p^0 + \Lambda^0_{\ 1} p^1)(\Lambda^0_{\ 0} q^0 + \Lambda^0_{\ 1} q^1) - (\Lambda^1_{\ 0} p^0 + \Lambda^1_{\ 1} p^1)(\Lambda^1_{\ 0} q^0 + \Lambda^1_{\ 1} q^1) \\
&\quad -(p^0 q^0 - p^1 q^1) \\
&= (\Lambda^0_{\ 0}\Lambda^0_{\ 0} - \Lambda^1_{\ 0}\Lambda^1_{\ 0} - 1) p^0 q^0 + (\Lambda^0_{\ 0}\Lambda^0_{\ 1} - \Lambda^1_{\ 0}\Lambda^1_{\ 1}) p^0 q^1 \\
&\quad +(\Lambda^0_{\ 1}\Lambda^0_{\ 0} - \Lambda^1_{\ 1}\Lambda^1_{\ 0}) p^1 q^0 + (\Lambda^0_{\ 1}\Lambda^0_{\ 1} - \Lambda^1_{\ 1}\Lambda^1_{\ 1} + 1) p^1 q^1
\end{aligned}
$$

is solved by

$$\Lambda^0_{\ 0} = \Lambda^1_{\ 1} = \pm \cosh \eta \qquad \Lambda^0_{\ 1} = \Lambda^1_{\ 0} = \mp \sinh \eta \ ,$$

where η is the ''rapidity'' of the boost. The usual choice is the upper sign.

How can we relate η to the relative velocity v between O and O'?

- Let us take two events and describe them in O and O':
  - A: the origins of O and O' overlap; set $t = t' = 0$
  - B: at the origin of O' after the time t', where $t = \Delta t$

2. Special Relativity (SR) — Lorentz transformations

determining Boosts ... continued

- The coordinates of A are $a^\mu = a'^\mu = (0, 0, 0, 0)$
- The coordinates of B
  - in O are $b^\mu = (\Delta t, v\Delta t, 0, 0)$ because O' was moving with the constant relative velocity v for the time Δt
  - in O' are $b'^\mu = (t', 0, 0, 0)$ because B is at the origin of O'
- But
$$\begin{aligned} b'^\mu &= \Lambda^\mu{}_\nu b^\nu \\ &= (\cosh\eta\,\Delta t - \sinh\eta\,v\Delta t\,,\ -\sinh\eta\,\Delta t + \cosh\eta\,v\Delta t\,,\ 0\,,\ 0) \end{aligned}$$

  Therefore
$$\begin{aligned} t' &= \cosh\eta\,\Delta t - \sinh\eta\,v\Delta t \\ 0 &= -\sinh\eta\,\Delta t + \cosh\eta\,v\Delta t \end{aligned}$$

  or
$$v = \frac{\sinh\eta}{\cosh\eta} = \tanh\eta \sim \eta \quad \text{for } \eta \text{ small}$$

Lorentz transformations on vectors

- A vector V^μ can be understood as the distance of two events

  $\Rightarrow$ Its transformation is the same as for events

- We used already the coordinate representation of events

$$\Rightarrow \qquad V'^\mu = \Lambda^\mu{}_\nu V^\nu$$

- If we want to write Λ as a matrix
  - we have to choose how we represent the vector V^μ.
  - the usual representation is a column-vector:

$$V^\mu = (V^0, V^1, V^2, V^3)^\top = \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

  - then we can write the Lorentz transformation as

$$\begin{pmatrix} V'^0 \\ V'^1 \\ V'^2 \\ V'^3 \end{pmatrix} = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} \doteq \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}$$

2. Special Relativity (SR) — Lorentz transformations

connecting to "conventional" Lorentz transformations

- Lorentz transformations are usually written down using equations:

$$\begin{aligned} t' &= \gamma(t - \tfrac{v\cdot x}{c^2}) \\ x' &= \gamma(x - v \cdot t) \\ y' &= y \\ z' &= z \end{aligned} \qquad \text{or better:} \qquad \begin{aligned} ct' &= \gamma(ct - \tfrac{v}{c} \cdot x) \\ x' &= \gamma(x - \tfrac{v}{c} \cdot ct) \\ y' &= y \\ z' &= z \end{aligned}$$

  - where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

- defining $\beta = \frac{v}{c}$ we can easily connect to the matrix-form of Λ
  - suppressing the unchanged coordinates y and z:

$$\begin{aligned} ct' &= \gamma(\ \ ct - \beta \cdot x) \\ x' &= \gamma(-\beta \cdot t + x) \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \cdot \begin{pmatrix} ct \\ x \end{pmatrix}$$

- comparing to the matrix-form with the pseudo rapidity η
  - we see: $\cosh\eta = \gamma$ and $\sinh\eta = \gamma\beta$, or $\beta = \tanh\eta$

More on vectors, the metric, and Lorentz transformations

- We defined the scalar product of contravariant* vectors:

  * contravariant can be understood as: the vector has an upper index

$$(p.q) = p^\mu q^\nu g_{\mu\nu} = p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3 \quad ,$$

  where $g_{\mu\nu} = g_{\nu\mu}$ is the metric with $g_{00} = 1$, $g_{ii} = -1$, and $g_{\mu \neq \nu} = 0$

- We can define covariant vectors with the index down: $V_\mu = g_{\mu\nu} V^\nu$
- The index can be raised again by $V^\mu = g^{\mu\nu} V_\nu$
- This obviously gives $g^{\mu\nu} g_{\nu\rho} = g^{\nu\mu} g_{\nu\rho} = g^{\mu\nu} g_{\rho\nu} = \delta^\mu_\rho$
- That means for the Lorentz transformations:

$$V'_\mu = g_{\mu\lambda} V'^\lambda = g_{\mu\lambda} \Lambda^\lambda{}_\kappa \, V^\kappa = g_{\mu\lambda} \Lambda^\lambda{}_\kappa \, g^{\kappa\nu} V_\nu = (\Lambda^\mu{}_\nu)^{-1} V_\nu$$

  or

$$(\Lambda^\mu{}_\nu)^{-1} = g_{\mu\lambda} \Lambda^\lambda{}_\kappa \, g^{\kappa\nu} = \Lambda_\mu{}^\nu$$

More on the Matrix Representation for Λ

- We can do the same trick (matrix representation) for covariant vectors
  - just $\Lambda(v)$ will look differently:

$$\Lambda(v)_\mu{}^\nu = g_{\mu\lambda}\Lambda(v)^\lambda{}_\kappa\, g^{\kappa\nu} \text{ "="} \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}$$

  - here we represent the covariant vectors also as column vectors!
    * in order to use the normal matrix multiplication: $(V'_\mu) = (\Lambda(v)_\mu{}^\nu) \cdot (V_\nu)$

- The matrix representation of $\Lambda(v)_\mu{}^\nu$ has the same form as $\Lambda(-v)^\mu{}_\nu$
- Both $g_{\mu\nu}$ and $g^{\mu\nu}$ can be written as diag$(1,-1,-1,-1)$, but they are not matrices in the same way as $\Lambda(v)_\mu{}^\nu$ or $\Lambda(-v)^\mu{}_\nu$ are matrices:

$\Lambda^\mu{}_\nu$	$\times$	contravariant vector	$\rightarrow$	contravariant vector
$\Lambda_\mu{}^\nu$	$\times$	covariant vector	$\rightarrow$	covariant vector
$g_{\mu\nu}$	$\times$	contravariant vector	$\rightarrow$	covariant vector
$g^{\mu\nu}$	$\times$	covariant vector	$\rightarrow$	contravariant vector

Lorentz transformations of fields

- Two observers, O and O', can agree on a space-time point x by calling it an event X
  - X might have different coordinates x^μ and x'^μ in O and O', but it is nevertheless the same point.
  - O and O' can compare the value of different fields at X
- The simplest field is the scalar field $\phi(x)$:

$$\phi'(X) = \phi(X)$$

- The vector fields $v^\mu(x)$ or $v_\mu(x)$ transform like a vectors:

$$v'^\mu(X) = \Lambda^\mu{}_\nu v^\nu(X) \qquad v'_\mu(X) = \Lambda_\mu{}^\nu v_\nu(X)$$

- Tensor fields $t^{\mu\nu}_{\rho\kappa\lambda}(x)$ transform like the product of vectors:

$$t'^{\mu\nu}_{\rho\kappa\lambda}(X) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda_\rho{}^\gamma \Lambda_\kappa{}^\delta \Lambda_\lambda{}^\epsilon t^{\alpha\beta}_{\gamma\delta\epsilon}(X)$$

Translation and Rotation Operators

- The momentum operator $\vec{P} = -i\frac{\partial}{\partial \vec{x}} = -i\vec{\partial}$ generates translations:
  - in index notation: $P_k = -i\frac{\partial}{\partial x^k} = -i\partial_k$

$$
\begin{aligned}
e^{ia^k P_k} f(x) &= e^{a^k \partial_k} f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(a^k\partial_k)^n f(x) \\
&= f(x) + a^k \partial_k f(x) + \frac{1}{2} a^j a^k \partial_j \partial_k f(x) + \dots
\end{aligned}
$$

  - the Taylorseries of $f(x+a)$ is

$$
f(x+a) = f(x) + a^k \partial_k f(x) + \frac{1}{2} a^j a^k \partial_j \partial_k f(x) + \dots = e^{i\vec{a}\vec{P}} f(x)
$$

$\Rightarrow$ the operator $e^{i\vec{a}\vec{P}}$ moves the function f by the amount $\vec{a}$

- The angular momentum operator $\vec{L} = \vec{X} \times \vec{P}$ generates rotations
  - in index notation: $L_j = \epsilon_{jk\ell}\, x^k P_\ell = -i\epsilon_{jk\ell}\, x^k \partial_\ell$
  - or $L_x = i(z\partial_y - y\partial_z)$, $L_y = i(x\partial_z - z\partial_x)$, $L_z = i(y\partial_x - x\partial_y)$

Translation and Rotation Operators

- The components of $\vec{L}$ do not commute:
  - if you rotate around the $\hat{x}$-axis and then around the $\hat{y}$-axis, you get a different result than rotating first around $\hat{y}$ and then $\hat{x}$.
  - mathematically:

$$\begin{aligned}
[L_y, L_x] &= i^2[(x\partial_z - z\partial_x)(z\partial_y - y\partial_z) - (z\partial_y - y\partial_z)(x\partial_z - z\partial_x)] \\
&= i^2[(x\partial_y + xz\partial_z\partial_y - xy\partial_z^2 - z^2\partial_x\partial_y + zy\partial_x\partial_z) \\
&\quad -(zx\partial_y\partial_z - z^2\partial_y\partial_x - yx\partial_z^2 + y\partial_x + yz\partial_z\partial_x)] \\
&= i^2[x\partial_y - y\partial_x] = -iL_z
\end{aligned}$$

  - or in index notation: $[L_j, L_k] = i\epsilon_{jk\ell}L_\ell$ $\Rightarrow$ Rotationgroup
- but the square $L^2 = \vec{L}\cdot\vec{L} = L_kL_k$ does commute:

$$\begin{aligned}
[L^2, L_j] &= L_k[L_k, L_j] + [L_k, L_j]L_k = L_k i\epsilon_{kj\ell}L_\ell + i\epsilon_{kj\ell}L_\ell L_k \\
&= L_h i\epsilon_{hjm}L_m + i\epsilon_{mjh}L_hL_m = i(\epsilon_{hjm} + \epsilon_{mjh})L_hL_m = 0
\end{aligned}$$

$\Rightarrow$ use L^2 and L_z to describe quantum mechanical states (particles)

Eigenstates of the Rotationgroup

- We write an eigenstate of the operators L^2 and L_z as $|\lambda, m\rangle$

$$L^2|\lambda, m\rangle = \lambda|\lambda, m\rangle \qquad \text{and} \qquad L_z|\lambda, m\rangle = m|\lambda, m\rangle$$

  - $|f\rangle$ is called a ket and used to denote a quantum mechanical state.

- We define the ladder operators $L_\pm = L_x \pm iL_y$ with

$$\begin{aligned} [L^2, L_\pm] &= [L^2, L_x] \pm i[L^2, L_y] = 0 \qquad \text{and} \\ [L_z, L_\pm] &= [L_z, L_x] \pm i[L_z, L_y] = iL_y \pm i(-iL_x) = \pm(L_x \pm iL_y) = \pm L_\pm \end{aligned}$$

$\Rightarrow$ $L_\pm|\lambda, m\rangle$ is also an eigenstate of L^2 and L_z :

$$\begin{aligned} L^2(L_\pm|\lambda, m\rangle) &= ([L^2, L_\pm] + L_\pm L^2)|\lambda, m\rangle = 0 + L_\pm L^2|\lambda, m\rangle \\ &= L_\pm\lambda|\lambda, m\rangle = \lambda(L_\pm|\lambda, m\rangle) \end{aligned}$$

and

$$\begin{aligned} L_z(L_\pm|\lambda, m\rangle) &= ([L_z, L_\pm] + L_\pm L_z)|\lambda, m\rangle = (\pm L_\pm + L_\pm L_z)|\lambda, m\rangle \\ &= (\pm L_\pm + L_\pm m)|\lambda, m\rangle = (m \pm 1)(L_\pm|\lambda, m\rangle) \end{aligned}$$

Eigenstates of the Rotationgroup

- $L_\pm$ does not change the eigenvalue λ of the state $|\lambda, m\rangle$
- $L_\pm$ changes the eigenvalue m of the state $|\lambda, m\rangle$

$\Rightarrow$ the states $|\lambda, m+n\rangle$ with $n \in Z$ are related

$\Rightarrow$ for each λ there would be ∞ many states unless there is

- $a = m_{\text{max}}$ with $L_+|\lambda, a\rangle = 0$ and
- $b = m_{\text{min}}$ with $L_-|\lambda, b\rangle = 0$

- using

$$\begin{aligned} L_\mp L_\pm &= (L_x \mp iL_y)(L_x \pm iL_y) \;=\; L_x^2 \pm iL_xL_y \mp iL_yL_x + L_y^2 \\ &= (L_x^2 + L_y^2 + L_z^2) - L_z^2 \pm i[L_x, L_y] \;=\; L^2 - L_z^2 \pm i(iL_z) \\ &= L^2 - L_z(L_z \pm 1) \end{aligned}$$

we can relate a and b

Eigenstates of the Rotationgroup

- relating a and b:
  - $0 = L_-L_+|\lambda, a\rangle = (\lambda - (a^2 + a))|\lambda, a\rangle \quad \Rightarrow \quad \lambda = a^2 + a$
  - $0 = L_+L_-|\lambda, b\rangle = (\lambda - (b^2 - b))|\lambda, b\rangle \quad \Rightarrow \quad \lambda = b^2 - b$

$$a(a+1) = b(b-1) \quad \text{or} \quad a = -b$$

- Applying (L_-) n times on the state $|\lambda, a\rangle$ gives $|\lambda, a - n\rangle$
- for some n we have to reach $|\lambda, b\rangle \quad \Rightarrow \quad a - n = b$
- with $a = -b$ we get $a - n = -a$ or $m_{\text{max}} = a = \frac{n}{2}$
- The rotationgroup allows for half integer eigenstates

$$\Rightarrow \qquad \text{Spinors}$$

- used to describe fermions: electron, proton, neutron, neutrino, ...