

1. Special Relativity (SR) — Introduction

Lectures

- Introduction, Invariants
- Lorentz transformations
- Algebra of the Lorentz group

Links

- Lecture notes by David Hogg: <http://cosmo.nyu.edu/hogg/sr/sr.pdf>
— or: <http://web.vu.lt/ff/t.gajdosik/wop/sr.pdf>
- Tatsu Takeuchi: <http://www.phys.vt.edu/~takeuchi/relativity/notes/>
- "Special Relativity for Particle Physics":
<http://web.vu.lt/ff/t.gajdosik/wop/sr4wop.pdf>

1. Special Relativity (SR) — Introduction

History

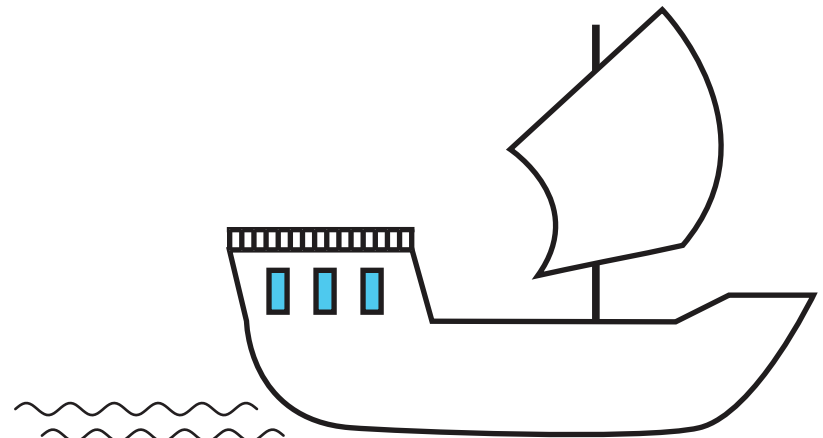
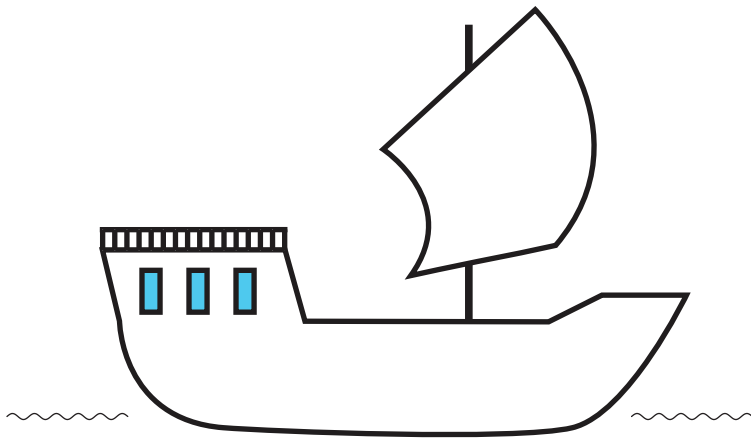
- 1632: Galileo Galilei describes the principle of relativity:
 - "Dialogue concerning the Two Chief World Systems"
- 1861: Maxwell's equations
- 1887: Michelson-Morley experiment
- 1889 / 1892: Lorentz – Fitzgerald transformation
- 1905: Albert Einstein publishes the Theory of Special Relativity:
 - "On the Electrodynamics of Moving Bodies"
- 1908: Hermann Minkovsky introduces 4D space-time

1. Special Relativity (SR) — Introduction

Galilean Invariance:

Every physical theory should mathematically look the same to every inertial observer

- for Galileo it was the mechanics and kinematics:
 - water dropping down
 - insects flying
 - throwing a ball or a stone
 - jumping around



1. Special Relativity (SR) — Introduction

Galilean Invariance / Galilean transformations: $t \rightarrow t'$, $\vec{x} \rightarrow \vec{x}'$

Two inertial observers, O and O' ,

- measure the same absolute time (i.e.: 1 second = 1 second')
 - Time translations : $t' = t + \tau$, $\vec{x}' = \vec{x}$
in index notation: $t' = t + \tau$, $x'_j = x_j$
- have at $t = 0$ a relative distance $\Delta\vec{r}$
 - Spatial translations : $t' = t$, $\vec{x}' = \vec{x} + \Delta\vec{r}$
in index notation: $t' = t$, $x'_j = x_j + \Delta r_j$
- have coordinate systems that are rotated by a relative rotation \mathbf{R}
 - Rotations : $t' = t$, $\vec{x}' = \mathbf{R} \cdot \vec{x}$, where \mathbf{R} is an orthogonal matrix
in index notation: $t' = t$, $x'_j = \mathbf{R}_{jk}x_k = \sum_{k=1}^3 \mathbf{R}_{jk}x_k$
- have a constant relative velocity \vec{v} (which can be zero, too)
 - Boosts : $t' = t$, $\vec{x}' = \vec{x} + \vec{v}t$
in index notation: $t' = t$, $x'_j = x_j + v_j t$

1. Special Relativity (SR) — Introduction

Galilean Group

- How Galilean transformations act on a quantum mechanical state
- What is a **group**?
 - a **set** with a **binary operation**:
 - an example is the set of numbers $\{0, 1, 2\}$ with the addition modulo 3 (i.e. taking only the remainder of the division by 3)
- Properties of a **group**
 - different transformations in the group do not give something that is outside the group
 - two transformations in different order give either zero or another transformation
- Each transformation depends on continuous parameters
 - The **Galilean Group** is a **Lie Group**

1. Special Relativity (SR) — Introduction

What's wrong with Galilean Invariance?

- Maxwell's equations describe the propagation of light depending on the electric permittivity and the magnetic permeability of the vacuum
- If the vacuum is the same for every inertial observer, he has to measure the same speed of light regardless, who emitted it
 - This is Einsteins second assumption!
- But then the addition of velocities described by the Galilean transformations are wrong
- Lorentz transformations describe correctly the measurements done regarding the speed of light
- Lorentz transformations include a transformation of the time, that the inertial observers measure
- Absolute time is a concept, that is not able to describe nature
 - That's wrong with the Galilean Invariance!

1. Special Relativity (SR) — Introduction

Axioms of Special Relativity

- Every physical theory should look the same mathematically to every inertial observer
- The speed of light in vacuum is independent from the movement of its emitting body

Consequences

- The speed of light in vacuum is maximal speed for any information
- The world has to be described by a 4D space-time: Minkowski space
- The simplest object is a scalar (field): $\phi(x)$
no structure except position and momentum
- The next simplest object is a spinor (field): $\psi^\alpha(x)$
a vector (field) can be described as a double-spinor

1. Special Relativity (SR) — Invariants

Vectors, Tensors, and notation

in the plane — i.e. in the 2D (Euclidean) space

- we can pick a coordinate system and describe points with coordinates
 - Cartesian coordinates (x, y)
 - Polar coordinates (r, θ)
- a vector can be understood as a difference of points
- position vector: difference between the position and the origin
- we can write the vector \vec{v}
 - as a row (v_x, v_y)
 - or as a column $\begin{pmatrix} v_x \\ v_y \end{pmatrix}$
 - or in index notation v_i or v^i , where we identify $v_x = v_1$ and $v_y = v_2$

1. Special Relativity (SR) — Invariants

Vectors, Tensors, and notation

multiplying vectors

- with a number, not a problem: $c * \vec{a} = (c * a_x, c * a_y)$
- with another vector: what do we want to get?
 - a number \Rightarrow scalar product: $\vec{a} \cdot \vec{b} := a_x * b_x + a_y * b_y$
 - another vector: there is no unique prescription ...
 - a tensor \Rightarrow tensor product: $\vec{a} \otimes \vec{b}$
 - * in index notation: $a_j \otimes b_k = a_j b_k = (a \otimes b)_{jk}$

what is a tensor?

- an object that looks like the tensor product of vectors ...
- easiest imaginable in indexnotation:
 - a tensor is an object with indices t_{jkl} or t^{jkl} or t^j_{kl}
- special tensors
 - a vector is a tensor of rank one: it has one index
 - a matrix is a tensor of rank two: it has two indices

1. Special Relativity (SR) — Invariants

Vectors, Tensors, and notation

multiplying tensors

- one index of each can be treated like a scalar product
 - ⇒ matrix multiplication
 - with $a = a_{jk}$ and $b = b_{mn}$: $a \cdot b = \sum_k a_{jk} * b_{kn}$
 - * here a and b can be understood as matrices
- in order to simplify the writing, we can omit the \sum symbol
 - ⇒ **Einsteins summation convention**
 - one sums over repeated indices: $a_{jk} * b_{kn} := \sum_k a_{jk} * b_{kn}$

index position can be used to distinguish objects

- example:
 - columnvector $\vec{v} = v^i = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$
 - rowvector $(\vec{v})^\top = v_i = (v_x, v_y)$
 - ★ then a matrix has to have upper and lower index!

1. Special Relativity (SR) — Invariants

Vectors, Tensors, and notation

in more dimensional space we just have more coordinates

- In 3D space (our 3D world):
 - $\vec{v} = (v_x, v_y, v_z) = v_i$ (in cartesian coordinates)
- In 4D Minkovsky space people do **not** write an arrow:
 - momentum $p = (E = p^t, p^x, p^y, p^z) = (p^0, p^1, p^2, p^3) = p^\mu$
 - * and the index is usually a greek letter: μ, ν, ρ , etc.
 - position $r = (ct, x, y, z) = (x^0, x^1, x^2, x^3) = r^\mu$
 - * time $ct = x^0$ is measured like spacial distances in meters
 - * The constant speed of light c is used as the conversion factor between seconds and meters

For the rest of the lecture we set $c = 1$. (i.e.: $3 \cdot 10^8 \text{m} = 1\text{s}$)

- so we measure time in seconds and distances in light-seconds (=300.000km)
- or distances in meters and time in "3 nanoseconds" (the time light needs to travel 1m)

1. Special Relativity (SR) — Invariants

What are invariant objects?

- Objects that are the same for every inertial observer
- Examples in 3D: rotations or translations
 - the distances ℓ between points: $\ell^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$
 - the angle α between directions: $\cos \alpha = (\vec{a} \cdot \vec{b}) / (|\vec{a}| * |\vec{b}|)$
- In 4D Minkovsky space: $(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$
 - The time t is measured like spacial distances in meters
 - The constant speed of light c is used as the conversion factor between seconds and meters

- Any **scalar** product of four-vectors in Minkovsky space:

$$(p \cdot q) = p^\mu q^\nu g_{\mu\nu} = p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3$$

1. Special Relativity (SR) — Invariants

What is the use of scalar products?

- Scalars are the same in every inertial frame
 - If one knows its value in one frame, one knows it in every frame
 - Use the most comfortable frame to calculate the value of a scalar!
- Events A and B happen at a certain time in a certain place:
 - In every frame they can be described by four-vectors
 $a^\mu = (a^0, a^1, a^2, a^3)$ and $b^\mu = (b^0, b^1, b^2, b^3)$
 - Their relative position $d^\mu = a^\mu - b^\mu$ is frame dependent
 - But their "4-distance" $d^2 = (d \cdot d)$ is invariant
 - d^2 classifies the causal connection of A and B

1. Special Relativity (SR) — Invariants

Classification of d^2

- If $d^2 > 0$ they are **time-like** separated:
 - one event happens before the other in every frame
 - there is a frame, where A and B happen at the same position
 - in this frame $d^\mu = (\Delta t, 0, 0, 0)$ with $\Delta t = \sqrt{d^2}$
- If $d^2 = 0$ they are **light-like** related. If $A \neq B$:
 - there is no frame, where A and B happen at the same time
 - there is no frame, where A and B happen at the same position
 - there is a frame, where $d^\mu = (\eta, \eta, 0, 0)$ with η arbitrary
- If $d^2 < 0$ they are **space-like** separated:
 - there is a frame, where A and B happen at the same time
 - in this frame $d^\mu = (0, \Delta s, 0, 0)$, with $\Delta s = \sqrt{-d^2}$,
if the x -axis is oriented in the direction \overline{AB}

1. Special Relativity (SR) — Invariants

Special scalar products

- Particles are described by their energy-momentum four-vector:

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p})$$

- The mass of the particle is defined in its **rest-frame**: $\vec{p} = 0$
- There, the energy-momentum four-vector is $p^\mu = (m, 0)$
- Since $p^2 = (p \cdot p)$ is a scalar, it is the same in every frame
- In the rest-frame $p^2 = m^2 - \vec{0}^2 = m^2$
- Therefore in **every frame**

$$m^2 = E^2 - \vec{p}^2 \quad !$$

- This can be applied to collisions, too: $(p_1 + p_2)^2$ is constant
 - In the **rest-frame** of $(p_1 + p_2)$ we have $\vec{p}_1 + \vec{p}_2 = 0 \Rightarrow (p_1 + p_2)^2 = (E_1 + E_2)^2$
 - * E_1 and E_2 are the energy values of p_1 and p_2 in the **rest-frame**!