

3. Special Relativity (SR) – Generators as differential operators

Translation and Rotation Operators

- The momentum operator $\vec{P} = -i\frac{\partial}{\partial \vec{x}} = -i\vec{\partial}$ generates translations:
 - in index notation: $P_k = -i\frac{\partial}{\partial x^k} = -i\partial_k$

$$\begin{aligned} e^{ia^k P_k} f(x) &= e^{a^k \partial_k} f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (a^k \partial_k)^n f(x) \\ &= f(x) + a^k \partial_k f(x) + \frac{1}{2} a^j a^k \partial_j \partial_k f(x) + \dots \end{aligned}$$

- the Taylorseries of $f(x + a)$ is

$$f(x + a) = f(x) + a^k \partial_k f(x) + \frac{1}{2} a^j a^k \partial_j \partial_k f(x) + \dots = e^{i\vec{a}\vec{P}} f(x)$$

⇒ the operator $e^{i\vec{a}\vec{P}}$ moves the function f by the amount \vec{a}

- The angular momentum operator $\vec{L} = \vec{X} \times \vec{P}$ generates rotations
 - in index notation: $L_j = \epsilon_{jkl} x^k P_l = -i\epsilon_{jkl} x^k \partial_l$
 - or $L_x = i(z\partial_y - y\partial_z)$, $L_y = i(x\partial_z - z\partial_x)$, $L_z = i(y\partial_x - x\partial_y)$

3. Special Relativity (SR) – Generators as differential operators

Translation and Rotation Operators

- The components of \vec{L} do not commute:
 - if you rotate around the \hat{x} -axis and then around the \hat{y} -axis, you get a different result than rotating first around \hat{y} and then \hat{x} .
 - mathematically:

$$\begin{aligned}[L_y, L_x] &= i^2[(x\partial_z - z\partial_x)(z\partial_y - y\partial_z) - (z\partial_y - y\partial_z)(x\partial_z - z\partial_x)] \\ &= i^2[(x\partial_y + xz\partial_z\partial_y - xy\partial_z^2 - z^2\partial_x\partial_y + zy\partial_x\partial_z) \\ &\quad - (zx\partial_y\partial_z - z^2\partial_y\partial_x - yx\partial_z^2 + y\partial_x + yz\partial_z\partial_x)] \\ &= i^2[x\partial_y - y\partial_x] = -iL_z\end{aligned}$$

- or in index notation: $[L_j, L_k] = i\epsilon_{jkl}L_l \Rightarrow$ **Rotationsgroup**
- but the square $L^2 = \vec{L} \cdot \vec{L} = L_k L_k$ does commute:

$$\begin{aligned}[L^2, L_j] &= L_k[L_k, L_j] + [L_k, L_j]L_k = L_k i\epsilon_{kjl}L_l + i\epsilon_{kjl}L_l L_k \\ &= L_h i\epsilon_{hjm}L_m + i\epsilon_{mjh}L_h L_m = i(\epsilon_{hjm} + \epsilon_{mjh})L_h L_m = 0\end{aligned}$$

\Rightarrow use L^2 and L_z to describe quantum mechanical states (particles)

3. Special Relativity (SR) – the Rotating group

Eigenstates of the Rotating group

- We write an eigenstate of the operators L^2 and L_z as $|\lambda, m\rangle$

$$L^2|\lambda, m\rangle = \lambda|\lambda, m\rangle \quad \text{and} \quad L_z|\lambda, m\rangle = m|\lambda, m\rangle$$

– $|f\rangle$ is called a **ket** and used to denote a quantum mechanical state.

- We define the ladder operators $L_{\pm} = L_x \mp iL_y$ with

$$[L^2, L_{\pm}] = [L^2, L_x] \mp i[L^2, L_y] = 0 \quad \text{and}$$

$$[L_z, L_{\pm}] = [L_z, L_x] \mp i[L_z, L_y] = iL_y \mp i(-iL_x) = \pm(L_x \mp iL_y) = \pm L_{\pm}$$

$\Rightarrow L_{\pm}|\lambda, m\rangle$ is also an eigenstate of L^2 and L_z :

$$\begin{aligned} L^2(L_{\pm}|\lambda, m\rangle) &= ([L^2, L_{\pm}] + L_{\pm}L^2)|\lambda, m\rangle = 0 + L_{\pm}L^2|\lambda, m\rangle \\ &= L_{\pm}\lambda|\lambda, m\rangle = \lambda(L_{\pm}|\lambda, m\rangle) \end{aligned}$$

and

$$\begin{aligned} L_z(L_{\pm}|\lambda, m\rangle) &= ([L_z, L_{\pm}] + L_{\pm}L_z)|\lambda, m\rangle = (\pm L_{\pm} + L_{\pm}L_z)|\lambda, m\rangle \\ &= (\pm L_{\pm} + L_{\pm}m)|\lambda, m\rangle = (m \pm 1)(L_{\pm}|\lambda, m\rangle) \end{aligned}$$

3. Special Relativity (SR) – the Rotatinggroup

Eigenstates of the Rotatinggroup

- L_{\pm} does not change the eigenvalue λ of the state $|\lambda, m\rangle$
- L_{\pm} changes the eigenvalue m of the state $|\lambda, m\rangle$

\Rightarrow the states $|\lambda, m + n\rangle$ with $n \in \mathbb{Z}$ are related

\Rightarrow for each λ there would be ∞ many states unless there is

* $a = m_{\min}$ with $L_-|\lambda, a\rangle = 0$ and

* $b = m_{\max}$ with $L_+|\lambda, b\rangle = 0$

- using

$$\begin{aligned}L_{\pm}L_{\mp} &= (L_x \mp iL_y)(L_x \pm iL_y) = L_x^2 \pm iL_xL_y \mp iL_yL_x + L_y^2 \\ &= (L_x^2 + L_y^2 + L_z^2) - L_z^2 \pm i[L_x, L_y] = L^2 - L_z^2 \pm i(iL_z) \\ &= L^2 - L_z(L_z \pm 1)\end{aligned}$$

we can relate a and b .

3. Special Relativity (SR) – the Rotating Group

Eigenstates of the Rotating Group

- relating a and b :

$$- 0 = L_+ L_- |\lambda, a\rangle = (\lambda - (a^2 + a)) |\lambda, a\rangle \quad \Rightarrow \quad \lambda = a^2 + a$$

$$- 0 = L_- L_+ |\lambda, b\rangle = (\lambda - (b^2 - b)) |\lambda, b\rangle \quad \Rightarrow \quad \lambda = b^2 - b$$

$$a(a + 1) = b(b - 1) \quad \text{or} \quad a = -b$$

- Applying (L_+) n times on the state $|\lambda, a\rangle$ gives $|\lambda, a + n\rangle$
- for some n we have to reach $|\lambda, b\rangle \quad \Rightarrow \quad a + n = b$
- with $a = -b$ we get $-b + n = b$ or $m_{\max} = b = \frac{n}{2}$
- The rotating group allows for half integer eigenstates

\Rightarrow Spinors

... we will encounter spinors again ...

3. Special Relativity (SR) – Algebra of the Poincaré group

Groups — what is a group?

- a set G together with a "multiplication \circ " with the properties:
 - for $a, b \in G \Rightarrow c = a \circ b \in G$
 - $(a \circ b) \circ c = a \circ (b \circ c)$
 - $\forall a \in G : \exists e \in G$ with $a \circ e = e \circ a = a$
 - $\forall a \in G : \exists a^{-1} \in G$ with $a \circ a^{-1} = a^{-1} \circ a = e$
- if $a \circ b = b \circ a \forall a, b \in G$: abelian group, otherwise non-abelian
 - abelian: $\{\mathcal{R}, +\}$ or $\{\mathcal{R}^+, \times\}$
 - non-abelian: regular square matrices with the matrix multiplication
- continuous groups: the elements depend on a continuous parameter
 - example: rotations around an axis $R[\theta]$ with $\theta \in [0, 2\pi)$
- Lie group: a continuous group with an analytic multiplication
 - $g[\vec{x}] \circ g[\vec{y}] = g[f(\vec{x}, \vec{y})]$ with $f(\vec{x}, \vec{y})$ analytic in \vec{x} and \vec{y}
 - the unit element is $e = g[\vec{0}]$

3. Special Relativity (SR) – Algebra of the Poincaré group

Lie groups and Lie algebras

- The $n \times n$ (complex) matrices form representations of Lie groups
- group multiplication is analytic \Rightarrow expansion around unit element
 - unit element $e = \mathbf{1}_{n \times n}$
 - representation $T(g[\alpha]) = \exp[i\alpha_i X_i] \Rightarrow X_k = -i \frac{\partial T(g[\alpha])}{\partial \alpha_k} \Big|_{\vec{\alpha}=0}$
 - generators $\{X_k\}$ span the representation of the Lie group
- the generators $\{X_k\}$ fulfill the Lie algebra $[X_j, X_k] = C_{jk}^\ell X_\ell$
 - with the antisymmetric structure constants $C_{jk}^\ell = -C_{kj}^\ell$
 - rank of the group: number of commuting generators
 - a Casimir operator commutes with all generators $\Rightarrow \propto e$
- the indices i, j, k, ℓ need not indicate single numbers!
 - for the generators we will have $X_i = X_{[mn]} = -X_{[nm]}$

3. Special Relativity (SR) – Algebra of the Poincaré group

Representations of the Lie group

- using the Jacobi identity

$$\begin{aligned} 0 &= [A, [B, C]] + [B, [C, A]] + [C, [A, B]] \\ &= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC \end{aligned}$$

we get for the structure constants

$$\begin{aligned} 0 &= C_{bc}^d [A, D] + C_{ca}^d [B, D] + C_{ab}^d [C, D] \\ &= C_{bc}^d C_{ad}^e + C_{ca}^d C_{bd}^e + C_{ab}^d C_{cd}^e = -(C_{ca}^d C_{db}^e - C_{cb}^d C_{da}^e) + C_{ab}^d C_{cd}^e \end{aligned}$$

- writing the structure constants as matrices $(M_k)_j^\ell = C_{jk}^\ell$ we have

$$0 = -[(M_a)_c^d (M_b)_d^e - (M_b)_c^d (M_a)_d^e] + C_{ab}^d (M_d)_c^e$$

or

$$[M_a, M_b] = C_{ab}^d M_d$$

⇒ the structure constants form the adjoint representation of the Lie group

3. Special Relativity (SR) – Algebra of the Poincaré group

Lie Algebra of the rotation group

- a rotation around the \hat{z} -axis by the angle θ is done by the matrix

$$R[\theta] = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{i\theta L_z}$$

– in index notation: $R[\theta]^j_k = \cos \theta (\delta_1^j \delta_k^1 + \delta_2^j \delta_k^2) - \sin \theta (\delta_1^j \delta_k^2 - \delta_2^j \delta_k^1) + \delta_3^j \delta_k^3$

- so the generator of the rotations, iL_z , is

$$iL_z = \left. \frac{\partial R[\theta]}{\partial \theta} \right|_{\theta=0} = \begin{pmatrix} -\sin \theta & -\cos \theta & 0 \\ \cos \theta & -\sin \theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big|_{\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- and similar

$$iL_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad iL_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- these rotations (incl. L_x and L_y) act on 3d column vectors $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

3. Special Relativity (SR) – Algebra of the Poincaré group

Lie Algebra of the rotation group

- with simple matrix multiplication we can see:

$$[iL_x, iL_y] = -iL_z \quad [iL_y, iL_z] = -iL_x \quad [iL_z, iL_x] = -iL_y$$

- or in index notation with $x = 1$, $y = 2$, and $z = 3$: $[L_j, L_k] = i\epsilon_{jkl}L_l$

- but there is a **smaller dimensional** realisation of the rotation group!
 - using the **Pauli matrices**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- one can define the Spin matrices $S_k = \frac{1}{2}\sigma_k$, which give

$$[S_j, S_k] = i\epsilon_{jkl}S_l$$

- these Spin matrices act on 2d complex column vectors $\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$
with $|\alpha|^2 + |\beta|^2 = 1 \quad \Rightarrow \quad$ **Spinors**

\Rightarrow **fundamental representation** of the **rotation group** $SU(2)$

3. Special Relativity (SR) – Algebra of the Poincaré group

Rotations of Spinors

- with simple matrix multiplication we can see for the Pauli matrices:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}_{2 \times 2}$$

- So the finite rotation of a spinor around the \hat{y} -axis is

$$\begin{aligned} R[\theta] &= e^{i\theta S_z} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\theta \frac{1}{2} \sigma_y)^n = \sum_{n=2m}^{\infty} \frac{1}{n!} (i\frac{\theta}{2})^n \sigma_y^n + \sum_{n=2m+1}^{\infty} \frac{1}{n!} (i\frac{\theta}{2})^n \sigma_y^n \\ &= \sum_n \frac{(-1)^n (\frac{\theta}{2})^{2n}}{(2n)!} (\sigma_y^2)^n + i \sum_n \frac{(-1)^n (\frac{\theta}{2})^{2n+1}}{(2n+1)!} (\sigma_y^2)^n \sigma_y \\ &= \cos \frac{\theta}{2} * \mathbf{1}_{2 \times 2} + i \sin \frac{\theta}{2} \sigma_y = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \end{aligned}$$

- acting on the spinor $\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

⇒ spinors rotate only with **half** of the **rotation angle** θ

3. Special Relativity (SR) – Algebra of the Poincaré group

Lorentz transformations (like Galilean transformations)

consist of Boosts and Rotations

- a boost in \hat{x} was done by

$$\Lambda(\eta)^\mu{}_\nu = \cosh \eta (\delta_0^\mu \delta_\nu^0 + \delta_1^\mu \delta_\nu^1) - \sinh \eta (\delta_0^\mu \delta_\nu^1 + \delta_1^\mu \delta_\nu^0) + \delta_2^\mu \delta_\nu^2 + \delta_3^\mu \delta_\nu^3$$

- a rotation between \hat{y} and \hat{z} can be done by

$$\Lambda(\theta)^\mu{}_\nu = \delta_0^\mu \delta_\nu^0 + \delta_1^\mu \delta_\nu^1 + \cos \theta (\delta_2^\mu \delta_\nu^2 + \delta_3^\mu \delta_\nu^3) - \sin \theta (\delta_2^\mu \delta_\nu^3 - \delta_3^\mu \delta_\nu^2)$$

- we obtain the generators for boosts with $-i \frac{\partial \Lambda(\eta)^\mu{}_\nu}{\partial \eta} \Big|_{\eta=0} =$

$$-i \sinh \eta (\delta_0^\mu \delta_\nu^0 + \delta_1^\mu \delta_\nu^1) + i \cosh \eta (\delta_0^\mu \delta_\nu^1 + \delta_1^\mu \delta_\nu^0) \Big|_{\eta=0} = i (\delta_0^\mu \delta_\nu^1 + \delta_1^\mu \delta_\nu^0)$$

- we obtain the generators for rotations with $-i \frac{\partial \Lambda(\theta)^\mu{}_\nu}{\partial \theta} \Big|_{\theta=0} =$

$$+i \sin \theta (\delta_2^\mu \delta_\nu^2 + \delta_3^\mu \delta_\nu^3) + i \cos \theta (\delta_2^\mu \delta_\nu^3 - \delta_3^\mu \delta_\nu^2) \Big|_{\theta=0} = i (\delta_2^\mu \delta_\nu^3 - \delta_3^\mu \delta_\nu^2)$$

3. Special Relativity (SR) – Algebra of the Poincaré group

- The other boosts go in \hat{y} or \hat{z} direction: $i(\delta_0^\mu \delta_\nu^i + \delta_i^\mu \delta_\nu^0)$,
or with the indices $0i$ down: $(M_{0i})^\mu{}_\nu = i(\delta_0^\mu (-g_{i\nu}) + \delta_i^\mu g_{0\nu})$.
- The other rotations go in $\hat{x}\hat{y}$ or $\hat{x}\hat{z}$ direction: $i(\delta_j^\mu \delta_\nu^k - \delta_k^\mu \delta_\nu^j)$,
or with the indices jk up: $(M_{jk})^\mu{}_\nu = i(\delta_j^\mu (-g_{k\nu}) - \delta_k^\mu (-g_{j\nu}))$.
- both generators have now the same form: $(M_{\alpha\beta})^\mu{}_\nu = -i(\delta_\alpha^\mu g_{\beta\nu} - \delta_\beta^\mu g_{\alpha\nu})$

- with $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$ we get

$$\Lambda(\omega)^\mu{}_\nu = \exp[i(M_{\alpha\beta}\omega^{\alpha\beta})^\mu{}_\nu] = \exp[(\delta_\alpha^\mu g_{\beta\nu} - \delta_\beta^\mu g_{\alpha\nu})\omega^{\alpha\beta}]$$

- these generators fulfill the Lie algebra of the Lorentz group:

$$[M_{\alpha\beta}, M_{\gamma\delta}]^\mu{}_\nu = i(g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\delta}M_{\beta\gamma} + g_{\beta\delta}M_{\alpha\gamma})^\mu{}_\nu$$

- unifying time and spatial translations $P_\mu = (H, P_i)$
- we get the rest of the Poincaré algebra:

$$[P_\mu, P_\nu] = 0 \quad \text{and} \quad [M_{\alpha\beta}, P_\mu] = i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)$$

3. Special Relativity (SR) – Algebra of the Poincaré group

Invariants of the Poincaré group

- obviously $[ab, c] = a[b, c] + [a, c]b = abc - acb + acb - cab = abc - cab$

$$[P_\mu, P^2] = [P_\mu, P_\nu]P^\nu + P^\nu[P_\mu, P_\nu] = 0$$

and

$$\begin{aligned}[M_{\alpha\beta}, P^2] &= g^{\mu\nu}[M_{\alpha\beta}, P_\mu]P_\nu + g^{\mu\nu}P_\mu[M_{\alpha\beta}, P_\nu] \\ &= g^{\mu\nu}i(g_{\alpha\mu}P_\beta - g_{\beta\mu}P_\alpha)P_\nu + g^{\mu\nu}P_\mu i(g_{\alpha\nu}P_\beta - g_{\beta\nu}P_\alpha) \\ &= -2i[P_\alpha, P_\beta] = 0 \quad .\end{aligned}$$

$\Rightarrow P^2 = m^2$ invariant is a consequence of the Poincaré algebra!

- Another invariant is W^2

– with the Pauli-Lubanski vector $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}M_{\nu\rho}P_\lambda$

$$\begin{aligned}[P_\kappa, W^\mu] &= \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}([P_\kappa, M_{\nu\rho}]P_\lambda + M_{\nu\rho}[P_\kappa, P_\lambda]) \\ &= \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}i(g_{\rho\kappa}P_\nu - g_{\nu\kappa}P_\rho)P_\lambda = 0\end{aligned}$$

\Rightarrow Particles can be characterised simultaneously by the eigenvalues of P^2 and W^2

3. Special Relativity (SR) – Algebra of the Poincaré group

Invariants of the Poincaré group

- the spin vector W^μ is orthogonal to P_μ :

$$(P.W) = P^\mu \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} M^{\nu\rho} P^\lambda = 0$$

- For a particle at rest: $P_\mu = (m, 0)$
 - so the eigenvalue of P^2 is m^2
 - $W_\mu = \frac{1}{2} m \epsilon_{\mu\nu\rho 0} M^{\nu\rho} = m(0, \vec{J})$
 - so $W^2 = -m^2 \vec{J}^2 = -m^2 s(s+1)$
 - so the eigenvalue of W^2 is $m^2 s(s+1)$
- For a massless particle $P_\mu = (\eta, \eta, 0, 0)$
 - we have $P^2 = (P.W) = W^2 = 0$
 - * so the eigenvalues of P^2 and W^2 are 0
 - but $0 = \lambda^2 P^2 - 2\lambda(P.W) + W^2 = (\lambda P - W)^2$
 - and therefore: $W^\mu = \lambda P^\mu$ with the helicity $\lambda = 0, \pm\frac{1}{2}, \pm 1, \dots$
 - * λ depends on the representation (i.e. the spin) of the particle

⇒ Particles are characterised by mass and spin !

3. Special Relativity (SR) – Algebra of the Poincaré group

Investigating the Lorentz group

- distinguishing again boosts and rotations

$$K_i = M_{0i} = -M^{0i} \quad \text{and} \quad J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \quad ,$$

the Lorentz algebra gives

$$[J_j, J_k] = i\epsilon_{jkl}J_l \quad , \quad [K_j, K_k] = -i\epsilon_{jkl}J_l \quad , \quad [J_j, K_k] = i\epsilon_{jkl}K_l$$

- defining

$$L_i = N_i = \frac{1}{2}(J_i + iK_i) \quad \text{and} \quad R_i = N_i^\dagger = \frac{1}{2}(J_i - iK_i)$$

one gets

$$[L_j, R_k] = 0 \quad , \quad [L_j, L_k] = i\epsilon_{jkl}L_l \quad , \quad [R_j, R_k] = i\epsilon_{jkl}R_l$$

- the Lorentz algebra is similar to $SU(2)_L \otimes SU(2)_R$!
- two invariants: $L_i L_i = n(n+1)$ and $R_i R_i = m(m+1)$
 - $J_i = L_i + R_i \quad \Rightarrow \quad \text{spin } j = n + m$

3. Special Relativity (SR) – Algebra of the Poincaré group

classifying particles

- according to the eigenstates (n, m) of $SU(2)_L \otimes SU(2)_R$
 - $(0, 0)$ is a scalar
 - $(\frac{1}{2}, 0)$ is the χ_α left-handed Weyl-spinor
 - $(0, \frac{1}{2})$ is the $\bar{\eta}^{\dot{\alpha}}$ right-handed Weyl-spinor
 - $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ is $\Psi = \begin{pmatrix} \chi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}$, the Dirac-spinor
 - $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$ is $(\chi^\mu \bar{\eta}) = \chi^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\eta}^{\dot{\alpha}}$, the spin-1 four-vector
- ⇒ in that sense is the spinor the square root of the vector
- under **Parity**: $J_i \xrightarrow{P} J_i, K_i \xrightarrow{P} -K_i, \Rightarrow L_i \xleftarrow{P} R_i, (n, m) \xleftarrow{P} (m, n)$
 - the scalar stays the same
 - $(\frac{1}{2}, 0) \xleftarrow{P} (0, \frac{1}{2})$, therefore $\chi_\alpha \xleftarrow{P} \bar{\eta}^{\dot{\alpha}}$
 - $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \xleftarrow{P} (0, \frac{1}{2}) \oplus (\frac{1}{2}, 0) = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$,
 - ⇒ so a Dirac-spinor stays a Dirac-spinor
 - the four-vector stays the same

3. Special Relativity (SR) – Algebra of the Poincaré group

classifying particles

- Charge conjugation also interchanges $SU(2)_L \Leftrightarrow SU(2)_R$
 - like Parity
- ⇒ the combined transformation CP leaves $SU(2)_L$ and $SU(2)_R$ invariant
 - but it still includes mathematically a complex conjugation
- Time reversal T is an antiunitary transformation
 - it includes a complex conjugation

any quantum field theory

- built from the representations of the Poincaré algebra
 - that means: scalars, spinors, vectors
- ⇒ has to be invariant under CPT