2. Special Relativity (SR) — explicit Lorentz transformations

Particles with $m > 0$ can always be seen as boosted from their rest frame

- in frame O we see the particle as $p^\mu = (E, \vec{p}) \doteq (E, p, 0, 0)$
- in its rest frame O' the particle is seen as $p'^\mu = (m, \vec{0})$
- the Lorentz transformation (LT) for p'^1 gives

$$0 = \Lambda^1{}_0\, E + \Lambda^1{}_1\, p = (-\sinh\eta)E + (\cosh\eta)p = \cosh\eta(p - E\tanh\eta)$$

- remembering $\tanh\eta = v/c := \beta$
  - and $\gamma = [1-\beta^2]^{-1/2} = [1-\tanh^2\eta]^{-1/2} = \cosh\eta$
- we get the Lorentz transformation in conventional form

$$\begin{aligned} t' &= \gamma(t - \beta x) &&= \frac{E}{m}\left(t - \frac{p}{E}x\right) = m^{-1}\,(E\,t - p\,x) \\ x' &= \gamma(x - \beta t) &&= \frac{E}{m}\left(x - \frac{p}{E}t\right) = m^{-1}\,(E\,x - p\,t) \end{aligned}$$

2. Special Relativity (SR) — explicit Lorentz transformations

photons have $m = 0$ and cannot have a rest frame

- a particle emits a photon with frequency f in $\hat{x}$ direction
- in frame O we see the particle as $p^\mu = (E, \vec{p}) \doteq (E, p, 0, 0)$
- in O', the rest frame of the particle, the photon has
  - the energy $E' = |k'| = \hbar f$
  - and the four momentum $k'^\mu = (k', k', 0, 0)$
- in frame O we see this photon as $k^\mu = (k, k, 0, 0)$
- the Lorentz transformation from O to O' gives

$$k' = \gamma(k - \beta k) = k\gamma(1 - \beta) = k\sqrt{\frac{(1-\beta)^2}{1-\beta^2}} = k\sqrt{\frac{1-\beta}{1+\beta}} = k\sqrt{\frac{E-p}{E+p}}$$

- this is called the Doppler effect

Lorentz transformations consist of

- boosts with $t' = \gamma(t - \beta x) \qquad x' = \gamma(x - \beta t)$
- and rotations with $t' = t \qquad \vec{x}' = \mathbf{R}_\theta \cdot \vec{x}$
- since LTs form a group, we can make a general LT
  - by performing consecutive "elementary" boosts and rotations

- example 1, without rotations:
  - a particle A of mass M, traveling in $\hat{x}$-direction with velocity v
  - decays into B_1 and B_2 of equal mass m, both traveling in $\hat{x}$-direction

- in our frame O we have $P^\mu = (E, p)$ with $v = p/E$
- in the restframe O' of A we have $P'^\mu = (M, 0) = (E_1, p_1) + (E_2, p_2)$
  - so $0 = p_1 + p_2$ or $p_2 = -p_1$
  - since B_1 and B_2 have equal mass $E_2 = E_1 = M/2$
  - so the LTs into the restframes of $B_{1,2}$ have

$$\beta_{1,2} = p_{1,2}/E_{1,2} = \pm\sqrt{1 - (\tfrac{2m}{M})^2} =: \pm\beta \qquad \text{and} \qquad \gamma_{1,2} = \tfrac{M}{2m}$$

- the LT from O' into the restframe of $B_{1,2}$ is

$$\Lambda'_{1,2} = \begin{pmatrix} \gamma_{1,2} & -\gamma_{1,2}\beta_{1,2} \\ -\gamma_{1,2}\beta_{1,2} & \gamma_{1,2} \end{pmatrix} = \frac{M}{2m} \begin{pmatrix} 1 & \mp\beta \\ \mp\beta & 1 \end{pmatrix}$$

- the LT from our frame O into O' is

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} = \frac{E}{M} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}$$

- so the LT from our frame O into the restframe of $B_{1,2}$ is $(\Lambda_{1,2})^{\mu}{}_{\nu} = (\Lambda)^{\mu}{}_{\rho}\,(\Lambda'_{1,2})^{\rho}{}_{\nu}$ or

$$\Lambda_{1,2} \quad = \quad \frac{E}{M}\frac{M}{2m} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \mp\beta \\ \mp\beta & 1 \end{pmatrix} = \frac{E}{2m} \begin{pmatrix} 1 \pm v\beta & -v \mp \beta \\ -v \mp \beta & 1 \pm v\beta \end{pmatrix}$$

- the four momenta of $B_{1,2}$ in their respective rest frames are $(m, 0)$, so in O they are $p^{\mu}_{1,2} = (\Lambda_{1,2})^{\mu}{}_{\nu}\,(m, 0)^{\nu}$ or $p_{1,2} = \frac{E}{2}(1 \pm v\beta, -v \mp \beta)$
  - from this we can deduce their velocities: $v_{1,2} = p_{1,2}/E_{1,2} = \frac{v \pm \beta}{1 \pm v\beta}$

- this is the velocity addition rule of Special Relativity

- example 2, with rotations:
  - the particle A of mass M, traveling in $\hat{x}$-direction with velocity v
  - decays into B_1 and B_2 of equal mass m
  - in the restframe O' of A, with $\hat{x}'||\hat{x}$ and $\hat{y}'||\hat{y}$
    * B_1 moves with angle θ' to the $\hat{x}$-direction in the $\hat{x}$-$\hat{y}$-plane
    * B_2 moves with angle $\varphi' = \pi + \theta'$ to the $\hat{x}$-direction in the $\hat{x}$-$\hat{y}$-plane

- the LT from O' into the restframe of $B_{1,2}$ has to include an additional rotation $\mathbf{R}_{\theta',\varphi'}$. Ignoring $\hat{z}$:

$$\mathbf{R}_{\theta',\varphi'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm c_{\theta'} & \pm s_{\theta'} \\ 0 & \mp s_{\theta'} & \pm c_{\theta'} \end{pmatrix}$$

  - One has to rotate first the boost direction into the $\hat{x}'$-axis
  - then one performs the boost in the direction of the new $\hat{x}'$-axis
  - and then one has to rotate the axes back in $O'_{1,2}$:

$$\Lambda'_{1,2} = \mathbf{R}^{-1}_{\theta',\varphi'} \cdot \Lambda'_{x;1,2} \cdot \mathbf{R}_{\theta',\varphi'} \qquad \text{where} \quad \Lambda'_{x;1,2} = \begin{pmatrix} \frac{M}{2m} & -\frac{M}{2m}\beta & 0 \\ -\frac{M}{2m}\beta & \frac{M}{2m} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- writing only a single angle for the rotation in the $\hat{x}$-$\hat{y}$-plane

$$
\begin{aligned}
\Lambda &= \mathbf{R}_\theta^{-1} \cdot \Lambda' \cdot \mathbf{R}_\theta \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta & -s_\theta & 0 \\ 0 & s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta & s_\theta & 0 \\ 0 & -s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\gamma\beta c_\theta & -\gamma\beta s_\theta & 0 \\ -\gamma\beta c_\theta & \gamma c_\theta^2 + s_\theta^2 & (\gamma-1) s_\theta c_\theta & 0 \\ -\gamma\beta s_\theta & (\gamma-1) s_\theta c_\theta & \gamma s_\theta^2 + c_\theta^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\gamma\beta c_\theta & -\gamma\beta s_\theta & 0 \\ -\gamma\beta c_\theta & 1 + (\gamma-1) c_\theta^2 & (\gamma-1) s_\theta c_\theta & 0 \\ -\gamma\beta s_\theta & (\gamma-1) s_\theta c_\theta & 1 + (\gamma-1) s_\theta^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Again ignoring $\hat{z}$:

- the LT from our frame O into O' is

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{E}{M} & -\frac{E}{M}v & 0 \\ -\frac{E}{M}v & \frac{E}{M} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- the LT from O into the restframe of $B_{1,2}$ is $(\Lambda_{1,2})^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\rho} (\Lambda'_{1,2})^{\rho}{}_{\nu}$

  ... complicated, but straight forward to calculate

- it is much simpler to apply the LTs onto the fourvector we are interested:

$$p'^{\mu}_{1,2} = (\Lambda'_{1,2})^{\mu}{}_{\nu} (m, 0, 0)^{\nu} = m\gamma(1, \mp\beta c_{\theta'}, \mp\beta s_{\theta'})^{\mu}$$

  and $p^{\mu}_{1,2} = \Lambda^{\mu}{}_{\nu} p'^{\nu}_{1,2}$ or with $\gamma = \frac{M}{2m}$

$$p_{1,2} = m\frac{M}{2m} \begin{pmatrix} \frac{E}{M} & -\frac{E}{M}v & 0 \\ -\frac{E}{M}v & \frac{E}{M} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mp\beta c_{\theta'} \\ \mp\beta s_{\theta'} \end{pmatrix} = \frac{M}{2} \begin{pmatrix} \frac{E}{M}(1 \pm v\beta c_{\theta'}) \\ -\frac{E}{M}(v \pm \beta c_{\theta'}) \\ \mp\beta s_{\theta'} \end{pmatrix}$$

  – from this we can read off the angles $\tan\theta = \frac{M\beta s_{\theta'}}{E(v+\beta c_{\theta'})}$ and $\tan\phi = \frac{M\beta s_{\theta'}}{E(-v+\beta c_{\theta'})}$

2. Special Relativity (SR) — projection onto 2D

in astronomy we have a "natural" coordinate system

- we see only the light that moves radially to us

  ⇒ we can only measure the angles of a spherical coordinate system

- for simplicity we can still use a cartesian system,
  - aligning one axes with our line of sight
    - we will use the $\hat{x}$-axis for our line of sight

  ⇒ light rays will always have the four vector $k^\mu = (k, k, 0, 0)$

- the general Lorentz transformation describes the motion to us
  - for a movement away from us, we should take $\beta \to -\beta$
    - then we have the same convention as David Hogg, Chapter 7

in order to compare our observation with the emission,
we have to Lorentz transform into the emitters system

- the general LT applied to the light ray $k^\mu = k(1,1,0,0)$ gives

$$k' = k \begin{pmatrix} \gamma & +\gamma\beta c_\theta & +\gamma\beta s_\theta \\ +\gamma\beta c_\theta & \gamma c_\theta^2 + s_\theta^2 & (\gamma-1)s_\theta c_\theta \\ +\gamma\beta s_\theta & (\gamma-1)s_\theta c_\theta & \gamma s_\theta^2 + c_\theta^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = k \begin{pmatrix} \gamma(1+\beta c_\theta) \\ \gamma c_\theta(c_\theta+\beta) + s_\theta^2 \\ \gamma s_\theta(c_\theta+\beta) - s_\theta c_\theta \end{pmatrix}$$

  – the energy emitted is $(k')^0 = k\gamma(1+\beta c_\theta)$

- astronomers define the dimensionless redshift z by

$$1 + z \equiv \frac{\Delta t_r}{\Delta \tau_e} = \frac{\text{emitted frequency}}{\text{received frequency}} = \frac{\text{emitted energy}}{\text{received energy}} = \gamma(1+\beta c_\theta)$$

  – which is nothing else, but the shift due to the Doppler effect

- when the object is moving to us, z is negative and called blueshift

2. Special Relativity (SR) — stellar abberation

When charting the sky

- we know, that the earth is moving relative to the "background"
  - like circling the sun or with the sun the Milky Way
- for simplicity we again ignore the $\hat{z}$-direction
  - then the "position" of the star is described by the angle θ:

$$k^\mu = k(1, c_\theta, s_\theta)$$

  - a non moving observer would see the star with the four vector

$$k'^\mu = \Lambda^\mu{}_\nu \, k^\nu = k(\gamma(1+\beta c_\theta), \gamma(c_\theta+\beta), s_\theta) = k'(1, c_{\theta'}, s_{\theta'})$$

  - the ratio k/k' is the discussed Doppler shift
- the change in the angle $\theta \to \theta'$ is called stellar abberation :

$$c_{\theta'} = \cos\theta' = \frac{\cos\theta + \beta}{1 + \beta\cos\theta} = \frac{c_\theta + \beta}{1 + \beta c_\theta}$$

2. Special Relativity (SR) — relativistic beaming

Brightness is defined as the observed radiation density: $I = \mathrm{d}E/\mathrm{d}t * (\mathrm{d}\Omega)^{-1}$

- I is independent of the distance R:
  - the observed amount of light goes down with R^{-2}
  - but the angular size goes down with R^{-2}, too.

- but I is not independent of the motion:
  - the moving object emitts light isotropically: $\mathrm{d}E'/\mathrm{d}t'$
    - we see the Doppler shift for the energy: $\mathrm{d}E' = \mathrm{d}E * \gamma(1 + \beta c_\theta)$
  - $1/\mathrm{d}t' \approx f'$ is the frequency of the emitted photons
  - this frequency f' is proportional to the energy of the photons E'
    - so a Doppler shifted frequency: $(1/\mathrm{d}t') = (1/\mathrm{d}t) * \gamma(1 + \beta c_\theta)$
  - as seen from stellar abberation
    - the perceived angle depends on the relative motion:

$$\cos\theta' = \frac{\cos\theta + \beta}{1 + \beta\cos\theta}$$

2. Special Relativity (SR) — relativistic beaming

- the solid angle $\mathrm{d}\Omega = \mathrm{d}\cos\theta * \mathrm{d}\phi$
  - $\mathrm{d}\phi$ is orthogonal to the direction of the boost
  - but $\mathrm{d}\cos\theta$ transforms:

$$\begin{aligned}\mathrm{d}\cos\theta' &= \mathrm{d}\left(\frac{\cos\theta+\beta}{1+\beta\cos\theta}\right) = \frac{\mathrm{d}\cos\theta}{1+\beta\cos\theta} - \frac{\cos\theta+\beta}{(1+\beta\cos\theta)^2}\beta\mathrm{d}\cos\theta \\ &= \frac{1+\beta\cos\theta-\beta\cos\theta-\beta^2}{(1+\beta\cos\theta)^2}\mathrm{d}\cos\theta = \frac{\mathrm{d}\cos\theta}{\gamma^2(1+\beta\cos\theta)^2}\end{aligned}$$

- the emitted brightness I' is

$$\begin{aligned} I' &= \frac{\mathrm{d}E'/\mathrm{d}t'}{\mathrm{d}\cos\theta' * \mathrm{d}\phi'} = \frac{\mathrm{d}E * \gamma(1+\beta c_\theta) * 1/\mathrm{d}t * \gamma(1+\beta c_\theta)}{\frac{\mathrm{d}\cos\theta}{\gamma^2(1+\beta\cos\theta)^2} * \mathrm{d}\phi} \\ &= \frac{\mathrm{d}E/\mathrm{d}t}{\mathrm{d}\cos\theta * \mathrm{d}\phi} * [\gamma(1+\beta c_\theta)]^4 = I * (1+z)^4\end{aligned}$$

  - when the object moves directly to us $c_\theta = -1$ and

$$\frac{I}{I'} = [\gamma(1-\beta)]^{-4} = \left(\frac{1+\beta}{1-\beta}\right)^2 \gg 1 \qquad \Rightarrow \quad \text{"beaming"}$$

2. Special Relativity (SR) — kinematic model — Milne universe

explosion in O' at $t' = 0$ and all fragments flying with constant velocity

- all positions are given by $\vec{r}' = \vec{v}'t'$
  - everything is moving away
  - the velocity is proportional to the distance

  $\Rightarrow$ Hubble flow

- our frame O is moving with one of the fragments
  - our time starts at the explosion with $t = 0$
  - each fragment came from the origin $(0, 0, 0, 0)$

  $\Rightarrow$ the worldline of each fragment goes through the origin
  - each fragment has a constant velocity $\vec{v}$

    $\Rightarrow \quad \vec{r} = \vec{v}t \quad \Rightarrow$ also Hubble flow

explosion in O' at $t' = 0$ and all fragments flying with constant velocity

- we see now, at t_0, another fragment
  - at the place, r_e away from us, where it emitted the light
- the fragment traveled the distance r_e from the Big Bang
  - for this distance it needed the time $t_e = r_e/v$
- the light traveled this distance r_e to us, needing the time r_e/c
- we see the light now at $t_0 = t_e + r_e/c = r_e(1/v + 1/c)$
- emitting the light, the fragments eigentime was $\tau^2 = (r_e/v)^2 - (r_e/c)^2$
- using $1 + z = t_0/\tau$ gives the angular diameter distance

$$d_A = r_e = ct_0 \frac{2z + z^2}{2(1+z)^2} \quad < \quad \frac{1}{2} ct_0$$

  - measured by the angular diameter, if the size is known
- knowing the intrinsic Luminosity $L = \int I \mathrm{d}\Omega$ and measuring the Flux
  - gives the luminosity distance $d_L = r = \sqrt{L/(4\pi F)}$
  - $d_L = d_A * (1+z)^{-4}$ in this kinematic model