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Links

- Lecture notes by David Hogg: `http://cosmo.nyu.edu/hogg/sr/sr.pdf`
  - or: `http://www.tfk.ff.vu.lt/~garfield/WoP/sr.pdf`

Galilean Invariance / Galilean transformations: $t \to t'$, $\vec{x} \to \vec{x}'$

Two inertial observers, O and O',

- measure the same absolute time (i.e.: 1 second = 1 second$'$).
  - Time translations : $t' = t + \tau$, $\vec{x}' = \vec{x}$

    in index notation: $t' = t + \tau$, $x'_j = x_j$
- have at $t = 0$ a relative distance $\Delta\vec{r}$.
  - Spatial translations : $t' = t$, $\vec{x}' = \vec{x} + \Delta\vec{r}$

    in index notation: $t' = t$, $x'_j = x_j + \Delta r_j$
- have coordinate systems that are rotated by a relative rotation $\mathbf{R}$.
  - Rotations : $t' = t$, $\vec{x}' = \mathbf{R} \cdot \vec{x}$, where $\mathbf{R}$ is an orthogonal matrix

    in index notation: $t' = t$, $x'_j = \mathbf{R}_{jk} x_k = \sum_{k=1}^{3} \mathbf{R}_{jk} x_k$
- have a constant relative velocity $\vec{v}$, which can be zero, too.
  - Boosts : $t' = t$, $\vec{x}' = \vec{x} + \vec{v}t$

    in index notation: $t' = t$, $x'_j = x_j + v_j t$

Galilean Group

- How the Galilean transformations act on a quantum mechanical state.
- What is a group?
  - a set with a binary operation:
  - an example is the set of numbers $\{0, 1, 2\}$ with the addition modulo 3 (i.e. taking only the remainder of the division by 3).
- Properties of a group
  - different transformations in the group do not give something that is outside the group.
  - two transformations in different order give either zero or another transformation.
- Each transformation depends on continuous parameters
  - The Galilean Group is a Lie Group.

What's wrong with Galilean Invariance?

- Maxwell's equations describe the propagation of light depending on the electric permittivity and the magnetic permeability of the vacuum.
- If the vacuum is the same for every inertial observer, he has to measure the same speed of light regardless, who emitted it.
  - This is Einsteins second assumption!
- But then the addition of velocities described by the Galilean transformations are wrong.
- Lorentz transformations describe correctly the measurements done regarding the speed of light.
- Lorentz transformations include a transformation of the time, that the inertial observers measure.
- Absolut time is a concept, that is not able to describe nature.
  - That's wrong with the Galilean Invariance!

Axioms of Special Relativity

- Every physical theory should look the same mathematically to every inertial observer.
- The speed of light in vacuum is independent from the movement of its emmitting body.

Consequences

- The speed of light in vacuum is maximum speed for any information.
- The world has to be described by a 4D space-time: Minovsky space.
- The simplest object is a scalar (field): $\phi(x)$
  no structure except position and momentum.
- The next simplest object is a spinor (field): $\psi^{\alpha}(x)$
  a vector (field) can be described as a double-spinor.

the plane — i.e. 2D (Euclidean) space

- we can pick a coordinate system and describe points with coordinates
  - Cartesian coordinates (x, y)
  - Polar coordinates (r, θ)

- a vector can be understood as a difference of points
  - position vector: difference between the position and the origin

- we can write the vector $\vec{v}$
  - as a row (v_x, v_y) ... or as a column $\binom{v_x}{v_y}$
  - in index notation v_i or v^i, where we identify $v_x = v_1$ and $v_y = v_2$

We can understand the plane as being generated by two vectors :

$$\text{plane} = \text{point} + a\hat{x} + b\hat{y} \qquad \text{with } a, b \in \mathcal{R}$$

- $\hat{x}$ and $\hat{y}$ are said to span the plane, which is a vectorspace

multiplying vectors

- with a number, not a problem: $c * \vec{a} = (c * a_x, c * a_y)$
- with another vector: what do we want to get?
  - a number $\Rightarrow$ scalar product: $\vec{a} \cdot \vec{b} := a_x * b_x + a_y * b_y$
  - another vector: there is no unique prescription . . .
  - a tensor $\Rightarrow$ tensor product: $\vec{a} \otimes \vec{b}$
    - in index notation: $a_j \otimes b_k = a_j b_k = (a \otimes b)_{jk}$

Geometric Algebra defines the geometric product of vectors :

for vectors $a = \vec{a}$, $b = \vec{b}$, etc.

- $a\, b$ has a symmetric and an antisymmetric part: $a\, b = a \cdot b + a \wedge b$
  - $a\, a = a \cdot a + a \wedge a = a^2$ is a number
    (the normal scalar product of the vector with itself)
  - $b\, a = b \cdot a + b \wedge a = a \cdot b - a \wedge b$ is a number plus a bivector
- $C = a\, b$ is called a multivector
- a multivector is NOT a tensor !

what is a tensor?

- an object that looks like the tensor product of vectors . . .
  - it transforms like a tensor product of vectors would transform
- easiest imaginable in indexnotation:
  - a tensor is an object with indices $t_{jk\ell}$ or $t^{jk\ell}$ or $t^{j}{}_{k\ell}$
- special tensors
  - a vector is a tensor of rank one: it has one index
  - a matrix is a tensor of rank two: it has two indices

tensors form also a vectorspace

- multiplying a tensor with a number gives again a tensor
  - the resulting tensor is of the same dimensions as the initial one.
- adding tensors of the same dimension gives again a tensor

multiplying tensors

- one index of each can be treated like a scalar product
  $\Rightarrow$ matrix multiplication
  - with $a = a_{jk}$ and $b = b_{mn}$: $(a \cdot b)_{jn} = \sum_k a_{jk} * b_{kn}$
    * here a and b can be understood as matrices

- in order to simplify the writing, we can omit the $\sum$ symbol
  $\Rightarrow$ Einsteins summation convention
  - one sums over repeated indices: $a_{jk} * b_{kn} := \sum_k a_{jk} * b_{kn}$

index position can be used to distinguish objects

- example:
  - columnvector $\vec{v} = v^i = \binom{v_x}{v_y}$
  - rowvector $(\vec{v})^\top = v_i = (v_x, v_y)$

- then a matrix has to have upper and lower index: $a^j{}_k \neq a_k{}^j$

in more dimensional space we just have more coordinates

- In 3D space (our 3D world):
  - $\vec{v} = (v_x, v_y, v_z) = v_i$ (in cartesian coordinates)

- In 4D Minkovsky space people do **not** write an arrow:
  - momentum $p = (E = p^t, p^x, p^y, p^z) = (p^0, p^1, p^2, p^3) = p^\mu$
    * and the index is usually a greek letter: μ, ν, ρ, etc.
  - position $r = (ct, x, y, z) = (x^0, x^1, x^2, x^3) = r^\mu$
    * time $ct = x^0$ is measured like spacial distances in meter.
    * The constant speed of light c is used as the conversion factor between seconds and meters.

For the rest of the lecture we set $c = 1$. (i.e.: $3 \cdot 10^8\text{m} = 1\text{s}$)

- so we measure time in seconds and distances in light-seconds
- or distances in meters and time in 3 nanometers.

What are invariant objects?

- Objects that are the same for every inertial observer.
- Examples in 3D: rotations or translations
  - the distances ℓ between points: $\ell^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$.
  - the angle α between directions: $\cos\alpha = (\vec{a} \cdot \vec{b})/(|\vec{a}| * |\vec{b}|)$.
- In 4D Minkovsky space: $(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$.
  - The time t is measured like spacial distances in meter.
  - The constant speed of light c is used as the conversion factor between seconds and meters.
- Any scalar product of four-vectors in Minkovsky space:

$$(p.q) = p^\mu q^\nu g_{\mu\nu} = p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3 \quad .$$

What is the use of scalar products?

- Scalars are the same in every inertial frame.
  - If one knows its value in one frame, one knows it in every frame.
  - ⇒ Use the most comfortable frame to calculate the value of a scalar!

- Events A and B happen at a certain time in a certain place:
  - In every frame they can be described by four-vectors $a^\mu = (a^0, a^1, a^2, a^3)$ and $b^\mu = (b^0, b^1, b^2, b^3)$.
  - Their relative position $d^\mu = a^\mu - b^\mu$ is frame dependent.
  - But their "4-distance" $d^2 = (d \cdot d)$ is invariant.
  - d^2 classifies the causal connection of A and B.

Classification of d^2

- If $d^2 > 0$ they are time-like separated:
  - one event happens before the other in every frame.
  - there is a frame, where A and B happen at the same position.
  - in this frame $d^\mu = (\Delta t, 0, 0, 0)$ with $\Delta t = \sqrt{d^2}$.
- If $d^2 = 0$ they are light-like related. If $A \neq B$:
  - there is no frame, where A and B happen at the same time.
  - there is no frame, where A and B happen at the same position.
  - there is a frame, where $d^\mu = (\eta, \eta, 0, 0)$ with η arbitrary.
- If $d^2 < 0$ they are space-like separated:
  - there is a frame, where A and B happen at the same time.
  - in this frame $d^\mu = (0, \Delta s, 0, 0)$, with $\Delta s = \sqrt{-d^2}$, if the x-axis is oriented in the direction $\overline{AB}$.

A special scalar product

- Particles are usually described by their energy-momentum four-vector:

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p})$$

  - The mass of the particle is defined in its rest-frame: $\vec{p} = 0$.
  - There, the energy-momentum four-vector is $p^\mu = (m, 0)$.
  - Since $p^2 = (p \cdot p)$ is a scalar, it is the same in every frame.
  - In the rest-frame $p^2 = m^2$.
  - Therefore in every frame

$$m^2 = E^2 - \vec{p}^2 \quad !$$

- This can be applied to collisions, too: $(p_1 + p_2)^2$ is constant.

Lorentz transformations

- relate the coordinate systems of two inertial observers.
- leave the "4-distance" invariant.
- assuming linearity, they can be written as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu} \quad .$$

  - These are called inhomogeneous Lorentz transformations (Λ, a).

Homogeneous Lorentz transformations have $a^{\mu} = 0$.

- They leave scalar products invariant: $(p'.q') = (p.q)$.
- They describe 3 Rotations and 3 Boosts
(cf. the Galilean transformations).

Rotations are the same as in the Galilean transformations.

For Boosts between O and O' let us align the coordinate systems:

- The origins of O and O' should be at the same place at $t = t' = 0$.
- The constant relative velocity v between O and O' should point in the $\hat{x}$-direction for both, O and O'.
- $\hat{y}$ and $\hat{z}$ should point in the same direction for both: $y' = y$ and $z' = z$.
- Only $ct = x^0$ and $x = x^1$ are affected by such a boost: $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}$ for either μ or ν being 2 or 3.
- So with $p' = \Lambda p$ and $q' = \Lambda q$ we have $(p'.q') - (p.q) = 0$.
- Since $y' = y$ and $z' = z$ we can ignore $\hat{y}$ and $\hat{z}$ in the equation

$$0 = (p'.q') - (p.q) = (p'^0 q'^0 - p'^1 q'^1) - (p^0 q^0 - p^1 q^1) \ .$$

Determining Boosts

$$\begin{aligned} 0 &= (\Lambda^0_0 p^0 + \Lambda^0_1 p^1)(\Lambda^0_0 q^0 + \Lambda^0_1 q^1) - (\Lambda^1_0 p^0 + \Lambda^1_1 p^1)(\Lambda^1_0 q^0 + \Lambda^1_1 q^1) \\ &\quad -(p^0 q^0 - p^1 q^1) \\ &= (\Lambda^0_0 \Lambda^0_0 - \Lambda^1_0 \Lambda^1_0 - 1) p^0 q^0 + (\Lambda^0_0 \Lambda^0_1 - \Lambda^1_0 \Lambda^1_1) p^0 q^1 \\ &\quad + (\Lambda^0_1 \Lambda^0_0 - \Lambda^1_1 \Lambda^1_0) p^1 q^0 + (\Lambda^0_1 \Lambda^0_1 - \Lambda^1_1 \Lambda^1_1 + 1) p^1 q^1 \end{aligned}$$

is solved by

$$\Lambda^0_0 = \Lambda^1_1 = \pm \cosh \eta \qquad \Lambda^0_1 = \Lambda^1_0 = \mp \sinh \eta \ ,$$

where η is the "rapidity" of the boost. The usual choice is the upper sign.

How can we relate η to the relative velocity v between O and O'?

- Let us take two events and describe them in O and O':
  - A: the origins of O and O' overlap; set $t = t' = 0$.
  - B: at the origin of O' after the time t', where $t = \Delta t$.

- The coordinates of A are $a^\mu = a'^\mu = (0,0,0,0)$.
- The coordinates of B
  - in O are $b^\mu = (\Delta t, v\Delta t, 0, 0)$ because O' was moving with the constant relative velocity v for the time Δt.
  - in O' are $b'^\mu = (t', 0, 0, 0)$ because B is at the origin of O'.
- But $b'^\mu = \Lambda^\mu{}_\nu b^\nu = (\cosh\eta\,\Delta t - \sinh\eta\, v\Delta t, -\sinh\eta\,\Delta t + \cosh\eta\, v\Delta t, 0, 0)$. Therefore

$$\begin{aligned} t' &= \quad \cosh\eta\,\Delta t - \sinh\eta\, v\Delta t \\ 0 &= -\sinh\eta\,\Delta t + \cosh\eta\, v\Delta t \end{aligned}$$

or

$$v = \frac{\sinh\eta}{\cosh\eta} = \tanh\eta \sim \eta \quad \text{for } \eta \text{ small.}$$

Lorentz transformations on vectors

- Each vector V^μ can be understood as the distance of two events.

- Its transformation is the same as for events in different inertial frames:

$$V'^\mu = \Lambda^\mu{}_\nu V^\nu \quad .$$

- Since $(V.W)$ is a scalar, $(V'.W') = (V.W)$:

$$V'^\mu W'_\mu = \Lambda^\mu{}_\nu V^\nu W'_\mu = V^\nu W_\nu \quad .$$

- So $\Lambda^\mu{}_\nu W'_\mu = W_\nu$ or $W'_\mu = (\Lambda^\mu{}_\nu)^{-1} W_\nu$

- What is now the inverse $(\Lambda(v))^{-1}$?
  - Obviously it should be $\Lambda(-v)$.

More on vectors, the metric, and Lorentz transformations

- We defined the scalar product of contravariant vectors:

$$(p.q) = p^\mu q^\nu g_{\mu\nu} = p^0 q^0 - p^1 q^1 - p^2 q^2 - p^3 q^3 \quad ,$$

  where $g_{\mu\nu} = g_{\nu\mu}$ is the metric with $g_{00} = 1$, $g_{ii} = -1$, and $g_{\mu\neq\nu} = 0$.

- We can define covariant vectors with the index down: $V_\mu = g_{\mu\nu} V^\nu$.

- The index can be raised again by $V^\mu = g^{\mu\nu} V_\nu$.

- This obviously gives $g^{\mu\nu} g_{\nu\rho} = g^{\nu\mu} g_{\nu\rho} = g^{\mu\nu} g_{\rho\nu} = \delta^\mu_\rho$.

- That means for the Lorentz transformations:

$$V'_\mu = g_{\mu\lambda} V'^\lambda = g_{\mu\lambda} \Lambda^\lambda{}_\kappa V^\kappa = g_{\mu\lambda} \Lambda^\lambda{}_\kappa g^{\kappa\nu} V_\nu = (\Lambda^\mu{}_\nu)^{-1} V_\nu$$

  or

$$(\Lambda^\mu{}_\nu)^{-1} = g_{\mu\lambda} \Lambda^\lambda{}_\kappa g^{\kappa\nu} = \Lambda_\mu{}^\nu \quad .$$

Lorentz transformations of fields

- Two observers, O and O', can agree on a space-time point x by calling it an event X.
  - X might have different coordinates x^μ and x'^μ in O and O', but it is nevertheless the same point.
  - O and O' can compare the value of different fields at that point X.

- The simplest field is the scalar field $\phi(x)$:

$$\phi'(X) = \phi(X) \quad .$$

- The vector fields $a^\mu(x)$ or $a_\mu(x)$ transform like vectors:

$$a'^\mu(X) = \Lambda^\mu{}_\nu a^\nu(X) \qquad a'_\mu(X) = \Lambda_\mu{}^\nu a_\nu(X) \quad .$$

- Tensor fields $t^{\mu\nu}_{\rho\kappa\lambda}(x)$ transform like the product of vectors:

$$t'^{\mu\nu}_{\rho\kappa\lambda}(X) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda_\rho{}^\gamma \Lambda_\kappa{}^\delta \Lambda_\lambda{}^\epsilon t^{\alpha\beta}_{\gamma\delta\epsilon}(X) \quad .$$