Measurements of the Cosmic Microwave Background Radiation

- First (involuntary) measurements by Penzias and Wilson in 1965
  ⇒ Nobel Prize in 1978

- COBE (Cosmic Background Explorer) is launched in 1989, takes data until 1991
  - FIRAS (Far Infrared Absolute Spectrophotometer)
    measures the frequency distribution in 1990
  ⇒ the CMB is a thermal blackbody radiation with $T \sim 2.725\, \text{K}$

  - DMR (Differential Microwave Radiometer)
    discovers the primary temperature anisotropy in 1992
  ⇒ Nobel Prize in 2006

- BOOMERanG and MAXIMA measure the acoustic oscillations in the angular power spectrum of the CMB anisotropy in 1999
How is the Cosmic Microwave Background Radiation measured?

- It is mainly a microwave radiation ⇒ radio antenna
  ⇒ directional measurements possible

- For higher accuracy in temperature differences
  ⇒ differential measurement
    * comparing the radiation coming from two different directions

- WMAP (Wilkinson Microwave Anisotropy Probe) measured
  - In 5 radio bands (23, 33, 41, 61, and 94 GHz with ~ 22% bandwidth)
  - 393,216 sky pixels with a solid angle of (0.77, 0.44, 0.26, 0.12, and 0.05) degree
    * each sky pixel is measured 1000 to 5000 times per year
7. General Relativity — CMB

WMAP

Precession rate: 1 rph
22.5° half-angle

A-side line of sight

MAP at L₂

1.5 x 10⁶ km

1.5 x 10⁸ km

Earth

Sun

Spin rate: 0.464 rpm

B-side line of sight

North Ecliptic Pole

South Ecliptic Pole

+90°

+45°

× -45°

-90°
7. General Relativity — Analysing the CMB

WMAP gives the temperature $T(\theta, \phi)$ of the CMB radiation

- the average temperature is
  \[ \langle T \rangle = \frac{1}{4\pi} \int T(\theta, \phi) \sin \theta \, d\theta \, d\phi = 2.725 \, \text{K} \]

- the temperature fluctuation
  \[ \frac{\delta T}{T}(\theta, \phi) = \frac{T(\theta, \phi) - \langle T \rangle}{\langle T \rangle} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi) \]

  can be described by spherical harmonics

- the spherical harmonics are orthonormal basis functions
  \[ Y_{\ell m}(\theta, \phi) = Ne^{im\phi}P_{\ell}^{m}(\cos \theta) \quad \text{with} \quad \int Y_{\ell m}^{*}Y_{\ell m'}^{*} \sin \theta \, d\theta \, d\phi = \delta_{\ell \ell'}\delta_{mm'} \]

  that satisfy an addition theorem
  \[ \sum_{m=-\ell}^{\ell} Y_{\ell m}^{*}Y_{\ell m}(\hat{n}_1)Y_{\ell m}(\hat{n}_2) = \frac{2\ell + 1}{4\pi} P_{\ell}(\cos \theta_{12}) \]

  with Legendre polynomial $P_{\ell}$ and the angle $\cos \theta_{12} = \hat{n}_1 \cdot \hat{n}_2$
multipoles of the CMB radiation

- the multipoles are given by
  \[ a_{\ell m} = \int Y^*_{\ell}(\theta, \phi) \delta^T_T(\theta, \phi) \sin \theta d\theta d\phi \]

- the two point correlation is
  \[ C(\theta_{12}) = \langle \delta^T_T(\hat{n}_1) \delta^T_T(\hat{n}_2) \rangle = \sum_{\ell_1, \ell_2, m_1, m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} \int Y^*_{\ell_1}(\hat{n}_1) Y_{\ell_2}(\hat{n}_2) \sin \theta d\theta d\phi \]

  - use Clebsch-Gordan coefficients to express the product of two Ys as a sum over single Ys
    \[ Y^*_{\ell_1}(\hat{n}_1) Y_{\ell_2}(\hat{n}_2) = |\ell_1 m_1 \rangle \otimes |\ell_2 m_2 \rangle = |\ell_1 m_1 \ell_2 m_2 \rangle \]

  - since we integrate over the angles \( \rightarrow \ell_3 = m_3 = 0 \)
    \( \Rightarrow \) \( m_1 \) and \( m_2 \) have to sum up to zero and \( \ell_1 = \ell_2 \)

  \( \Rightarrow \) only the "diagonal" terms contribute:
  \[ C(\theta_{12}) = \langle (\delta^T_T)^2 \rangle \]
7. General Relativity — Analysing the CMB

multipoles of the CMB radiation

• using the addition theorem (and $C_\ell = \sum_m |a_{\ell m}|^2$) we get

$$C(\theta_{12}) = \sum_{\ell,m} a_{\ell m} a^*_{\ell m} \int Y^*_{\ell m}(\hat{n}_1) Y_{\ell m}(\hat{n}_2) \sin \theta \, d\theta \, d\phi$$

$$= \sum_{\ell} C_\ell \frac{2\ell + 1}{4\pi} P_\ell(\cos \theta_{12})$$

• we are looking for the autocorrelation of density fluctuations
  – the angle between a direction and the same direction is zero
  $\Rightarrow \cos \theta_{12} = 1$ and $P_\ell(1) = 1$

• when dealing with a large sum, one can estimate it with an integral
  – in this case it is convenient to display the logarithm of $\ell$

$$C = \sum_{\ell=0}^\infty C_\ell \frac{2\ell + 1}{4\pi} P_\ell(1) \sim \int C_\ell \frac{\ell(2\ell + 1)}{4\pi} d(\ln \ell)$$

  – the interesting quantity is the integrand, more exactly $C_\ell$
How can we predict the multipoles of the CMB radiation?

- first we have to realize how the CMB is produced
  - the radiation left over from the hot big bang
- how exactly?
  - studying the distribution of photons
    * coming from the pair annihilations and scatterings
    * of the available particles during the expansion

⇒ coupled Einstein-Boltzmann equations
7. General Relativity — fluctuation of densities

the Boltzmann transport equation

\[ \frac{d}{dt} f_i(\vec{r}, \vec{p}, t) = \left( \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \vec{r} + \vec{F} \cdot \nabla \vec{p} \right) f_i(\vec{r}, \vec{p}, t) = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \]

- describes the change in the phase space density of particle $i$:
  - the flow $\vec{v}$ of particles changes their number in a region of space
  - the force $\vec{F}$ acting on the particles changes their momentum
  - the collisions (and decays) can change number and momentum

- how can we understand this equation in a covariant way?
  - we have $3 + 1$ and not only $3$ dimensions . . .

$\Rightarrow$ field equations (equations of motion) constrain $p^\mu$:
  - in flat space:

\[ m^2 = p^2 \quad \Rightarrow \quad p^0 = E = \sqrt{m^2 + \vec{p}^2} \]
7. General Relativity — fluctuation of densities in curved space

• field equations (equations of motion)
  – couple to the Einstein equations

• taking the Robertson-Walker metric as a background:

\[ g_{\mu\nu} \approx \begin{pmatrix} 1 & -a^2 \\ -a^2 & -a^2 \end{pmatrix} \]

– with scalar metric perturbations

\[ g_{00} = 1 - 2\Phi \quad \text{and} \quad g_{jk} = -\delta_{jk}a^2(1 - 2\Psi) \]

* \( \Phi \) corresponds to the Newtonian potential
* \( \Psi \) is the curvature perturbation

– one gets the constraint on the momentum

\[ m^2 = g_{\mu\nu}p^\mu p^\nu = E^2(1 - 2\Phi) - \vec{p}^2 a^2(1 - 2\Psi) \]

* that contains already a dependence on the curvature ...
7. General Relativity — fluctuation of densities

curvature is still given by Einsteins equations

- but now linearized
  - the scale factor $a$ is determined without perturbations
  - $\Phi$ and $\Psi$ are determined by the first order in the perturbations

- the stress energy tensor is given by the particles
  - weighted by their densities:
    \[ T^{\mu\nu} = \sum_{i=\text{all particles}} n_i T^{\mu\nu}_i \]
  - the density is given by the phase space density
    \[ n_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{r}, \vec{p}, t) \]
    * with $g_i$ the number of degrees of freedom per particle
    * which is also the zero-order "moment" of the phase space density
  - the velocity $\vec{v}_i$ is then the first-order moment
    \[ \vec{v}_i = \frac{g_i}{n_i} \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}_i}{E_i} f_i(\vec{r}, \vec{p}, t) = \left\langle \frac{\vec{p}_i}{E_i} \right\rangle \]
7. General Relativity — fluctuation of densities

the integrated collision term $C[f]$ is given by

$$C[f] = \int_{p_1} \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} = \int_{p_1} \int_{p_2} \int_{p_3} (2\pi)^4 \delta^4 (p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu) |\mathcal{M}|^2$$

$$\times \{ f_3 f_4 [1 \pm f_1][1 \pm f_2] - f_1 f_2 [1 \pm f_3][1 \pm f_4] \}$$

- with

$$\int_{p_i} := \int \frac{d^3 p_i}{(2\pi)^3 2 E_i}$$

- $\mathcal{M}$ describes the matrix element for the process $1 + 2 \leftrightarrow 3 + 4$
  - * to be calculated in Quantum Field Theory (next semester)
  - $\pm$ describes Bose enhancement / Pauli blocking ($+/ -$) for bosons / fermions
  - for high temperatures these factors become less important

$\Rightarrow$ the distributions $f = [e^{\frac{E - \mu}{kT}} \mp 1]^{-1}$ approach the Boltzmann distribution $e^{-\frac{E - \mu}{kT}}$
  - with the chemical potential $\mu$, which is related to the density

$$\frac{n_i}{g_i} = \int \frac{d^3 p}{(2\pi)^3} f_i(\vec{r}, \vec{p}, t) = e^{\mu/kT} \int \frac{d^3 p}{(2\pi)^3} e^{-E/kT} \approx \begin{cases} \left( \frac{m_i kT}{2\pi} \right)^{3/2} e^{-m_i/kT} & m_i \gg kT \\ \left( \frac{kT}{\pi^2} \right)^3 & m_i \ll kT \end{cases}$$

- in equilibrium $C[f] = 0 = (e^{(\mu_1 + \mu_2)/kT} - e^{(\mu_3 + \mu_4)/kT}) \int |\mathcal{M}|^2$

$\Rightarrow$ the chemical potentials have to be equal: $\mu_1 + \mu_2 = \mu_3 + \mu_4$
7. General Relativity — fluctuation of densities

introducing the equilibrium density \( n^{(0)}_i = g_i \int \frac{d^3 p}{(2\pi)^3} e^{-E/kT} \)

- one can rewrite \( e^{\mu/kT} = n_i/n_i^{(0)} \)
- and defined the thermally averaged cross section

\[
\langle v\sigma \rangle := \frac{(2\pi)^4}{n_1^{(0)} n_2^{(0)}} \int_{p_1} \int_{p_2} \int_{p_3} \int_{p_4} \delta^4(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu) |\mathcal{M}|^2 e^{-(E_1 + E_2)/kT}
\]

- then the Boltzmann equation for the number density becomes

\[
\int_{p_1} \frac{d}{dt} f_1(\vec{r}, \vec{p}, t) = \frac{d(a^3 n_1)}{a^3 dt} = n_1^{(0)} n_2^{(0)} \langle v\sigma \rangle \left\{ \frac{n_3^{(0)}}{n_1^{(0)}} \frac{n_4^{(0)}}{n_2^{(0)}} - \frac{n_1^{(0)}}{n_1^{(0)}} \frac{n_2^{(0)}}{n_2^{(0)}} \right\}
\]

- now \( \frac{d(a^3 n_1)}{a^3 dt} \sim H n_1 \) if \( H \ll n_2 \langle v\sigma \rangle \)

\[ \Rightarrow \text{the bracket has to become zero: } \frac{n_3^{(0)}}{n_1^{(0)}} \frac{n_4^{(0)}}{n_2^{(0)}} = \frac{n_1^{(0)}}{n_1^{(0)}} \frac{n_2^{(0)}}{n_2^{(0)}} \]

- chemical equilibrium . . . for heavy relics of the early universe
- nuclear statistical equilibrium . . . for Big Bang nucleosynthesis
- Saha equation . . . for recombination and ionization balance
applying this ansatz to

- dark matter particles and SM particles
  ⇒ dark matter abundance
    * needs non-equilibrium solution for freeze-out

- protons, neutrons and nuclei
  ⇒ Big Bang nucleosynthesis
    * needs non-equilibrium solution for neutron capture and decay

- electrons, positrons, photons, and neutrinos
  ⇒ CMB temperature versus neutrino temperature

- electrons, nuclei, and photons
  ⇒ recombination, CMB photon spectrum
    * still have to calculate the density fluctuations
  ⇒ we have to solve the linearized Einstein-Boltzmann equations
7. General Relativity — fluctuation of densities

linearized Einstein-Boltzmann equations

• introducing the metric perturbations $\Phi$ and $\Psi$

• introducing linearized density fluctuations for all particles:
  
  – for the photons $\Theta(\vec{x}, \hat{p}, t)$, independent of $|\hat{p}|$: 
    
    $$ f_\gamma(\vec{x}, \hat{p}, t) = \left[ \exp \left\{ \frac{|\hat{p}|}{T(t)[1 + \Theta(\vec{x}, \hat{p}, t)]} \right\} - 1 \right]^{-1} $$

  – for the other particles as a density contrast $\delta_i(\vec{x}, t)$ and a velocity $\vec{v}_i(\vec{x}, t)$:
    
    $$ n_i(\vec{x}, t) = n_i^{(0)}(\vec{x}, t) [1 + \delta_i(\vec{x}, t)] \text{ and } \vec{v}_i(\vec{x}, t) = \left\langle \frac{\vec{p}_i}{E_i} \right\rangle $$

  * both, $\delta_i(\vec{x}, t)$ and $\vec{v}_i(\vec{x}, t)$, are considered first order
  * the zero-order velocity is in equilibrium $\Rightarrow \vec{v}_i^{(0)} = 0$

– minimal amount of relevant particles:
  * photon * baryon (includes $e^- !$) * neutrino * dark matter

$\Rightarrow$ 10 coupled partial differential equations

  – Fourier transform $\Theta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \Theta(\vec{k})$ decouples the Fourier modes of $\Theta_\gamma$

  $\Rightarrow$ CMB power spectrum in the multipoles $C_\ell = \frac{1}{(-i)^{\ell}} \int_{-1}^{1} \frac{d(\hat{p} \cdot \hat{k})}{2} P_\ell(\hat{p} \cdot \hat{k}) \Theta(\vec{k})$

  – $\vec{v}_i$ is already the "dipole" of $n_i$
cosmic plasma at a temperature of 1 MeV (universe a few minutes old)

- in equilibrium: electrons, positrons, photons ... relativistic
- decoupled neutrinos ... still relativistic
  - for $e\nu \Leftrightarrow e\nu$ the coupling $\langle v\sigma \rangle < H$
- coupled baryons: protons and neutrons ... non-relativistic
  - antibaryons have annihilated with the baryons
  - the remaining baryons can come from Baryogenesis (introduced by Sakharov)
    * the baryon asymmetry is estimated as $(n_b - n_{\bar{b}})/s \approx 10^{-10}$
    * compatible with the ratio of baryons to photons today: $n_b/n_\gamma \approx 5.5 \cdot 10^{-10}$
  - these baryons can form nuclei ... step by step
  - coupled equations for all the elements until iron

- simplifications:
  - with $\sim 10^{10}$ photons per baryon and a temperature $kT \sim 0.1$ MeV
    * a binding energy of deuterium ($^2H$) of $E_b = 2.2$ MeV
    * there are $10^{10} \times e^{-2.2\text{MeV}/0.1\text{MeV}} \sim 2.2$ photons with $E_\gamma > E_b$ per nucleon
  - Li, Be, and B are less bound than He
  - nearly all nuclei are disintegrated by high energy photons
  - $^nH$ and $^n\text{He}$ are relevant, $^n\text{Li}$ only marginal
cosmic plasma at a temperature of 0.1 MeV (universe a few minutes old)

- for the reaction $p + n \rightleftharpoons D + \gamma$ we have the equilibrium

$$\frac{n_D}{n_p n_n} = \frac{n_D^{(0)}}{n_p^{(0)} n_n^{(0)}} = \frac{g_D \int \frac{d^3p}{(2\pi)^3} e^{-E_D/kT}}{g_p \int \frac{d^3p}{(2\pi)^3} e^{-E_p/kT} g_n \int \frac{d^3p}{(2\pi)^3} e^{-E_n/kT}}$$

$$\approx \frac{3}{2} \left[ \frac{m_p kT}{(2\pi)} \right]^{3/2} e^{-\frac{m_p}{kT}} \approx \frac{3}{4} \left[ \frac{2\pi m_D}{m_p m_n kT} \right]^{3/2} e^{\frac{m_p - m_n - m_D}{kT}}$$

- $m_p \simeq m_n \simeq m_D/2 \approx 1$ GeV, but $m_p + m_n - m_D = E_b \sim 2.2$ MeV
- at $kT \sim 1$ MeV the densities for $p$ and $n$ are similar to $n_b$

$$\frac{n_D}{n_b} = \frac{n_D}{n_p n_n} n_n \approx \frac{3}{4} \left[ \frac{2\pi m_D}{m_p m_n kT} \right]^{3/2} \frac{E_b}{e^{kT}} n_b n_\gamma \approx \frac{3}{4} \left[ \frac{4\pi}{m_p kT} \right]^{3/2} \frac{E_b}{e^{kT}} 5.5 \cdot 10^{-10} \frac{(kT)^3}{\pi^2}$$

$$\approx 1.86 \cdot 10^{-9} \left[ \frac{kT}{m_p} \right]^{3/2} \frac{E_b}{e^{kT}}$$

- which becomes smaller than 1 for $kT > 63.6$ keV

$\Rightarrow$ $n_D$ is at higher $kT$ exponentially suppressed: $\left. \frac{n_D}{n_b} \right|_{1\text{MeV}} \approx 5.3 \cdot 10^{-13}$
cosmic plasma at a temperature of 0.1 MeV (universe a few minutes old)

- the neutron-proton ratio comes mainly from $p + e \rightleftharpoons n + \nu$
  - for equilibrium we have $n_p^{(0)}/n_n^{(0)} = e^{(m_n-m_p)/kT} = e^{Q/kT}$
  - the electrons are still in thermal equilibrium: $n_e = n_e^{(0)}$
  - the neutrinos decouple with this reaction completely
  - starting with the Boltzmann equation
    \[
    \frac{d(a^3n_n)}{a^3dt} = n_n^{(0)}n_p^{(0)}\langle v\sigma \rangle \left\{ \frac{n_p^{(0)}}{n_p^{(0)}} - \frac{n_n^{(0)}}{n_n^{(0)}} \right\} = n_\nu^{(0)}\langle v\sigma \rangle \left\{ n_p e^{-Q/kT} - n_n \right\}
    \]
  - $\lambda_{np} = n_\nu^{(0)}\langle v\sigma \rangle$ describes the rate for neutron-proton conversion
  - the neutron fraction $X_n = \frac{n_n}{n_p + n_n}$ "freezes" below $kT \sim 0.5$ MeV
  - nucleosynthesis starts at $kT = 70$ keV with $X_n \approx 0.11$
    \[\Rightarrow\] He mass ratio $\sim 4 \cdot X_n/2 \approx 22%$
  - all neutrons are bound in He, since $E_b(^4\text{He}) \sim 28 \text{ MeV} \gg E_b(^2\text{D}) \sim 2.2 \text{ MeV}$
cosmic plasma at a temperature $< 0.1$ MeV (universe a few minutes old)

- only at $kT = 70$ keV nucleosynthesis really starts
  - nearly all D is processed further to $^4\text{He}$
  - the left over D depends on the baryon density $n_b$
    $\Rightarrow$ measurement of the ratio $D/H \sim 3 \cdot 10^{-5}$ determines $\Omega_b \sim 0.04$
  - the produced $^7\text{Li}$ gives also tight bounds
cosmic plasma at a temperature $< 14$ eV

- Recombination goes by the process $p + e \Leftrightarrow H + \gamma$
  - for equilibrium we have $\frac{n_en_p}{n_H} = \frac{n_e^{(0)}n_p^{(0)}}{n_H^{(0)}}$
  - with the free electron fraction $X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}$ we get
    \[
    \frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \left( \frac{m_e kT}{2\pi} \right)^{3/2} e^{(m_H - m_p - m_e)/kT}
    \]
  - for $kT \sim \epsilon_0 = m_p + m_e - m_H$ all Hydrogen is ionized
  - recombination has to end in an excited state
    - a photon from ground state recombination has $E_\gamma \geq \epsilon_0$
      \Rightarrow instant reionization
  - solving the equation for the electron fraction
    \Rightarrow determines the decoupling temperature (or redshift $\sim 1000$)
    \Rightarrow CMB pattern: $C_\ell$-distribution, CMB polarization
7. General Relativity — dark matter

dark matter balance

• for simplicity we take a single particle $X$
  - with a (very) weak coupling $X + X \leftrightarrow Y + Z$

• the Standard Model particles $Y$ and $Z$ are in thermal equilibrium
  $\Rightarrow n_{Y,Z} = n_{Y,Z}^{(0)}$ and the number density equation becomes

\[
\frac{d(a^3 n_X)}{a^3 dt} = \langle v \sigma \rangle \left\{ (n_X^{(0)})^2 - (n_X)^2 \right\}
\]

- eventually we want to express the density in terms of the temperature $kT$
  * the temperature scales inverse to the scale factor: $T \sim a^{-1}$
  * in the radiation dominated time $H(a) = H(a_1)(a_1/a)^2$
  * with $x = m_X a$ we have $\frac{dx}{dt} = \dot{a} = m H a = x H_m (x_m/x)^2 = H_m/x$
  * using $Y := a^3 n_X$ and $\frac{d}{dt} = \frac{dx}{dt} \frac{d}{dx} = \frac{H_m}{x} \frac{d}{dx}$ we get

\[
\frac{H_m dY}{x dx} = \frac{m_X^3}{x^3} \langle v \sigma \rangle \left( Y_{\text{EQ}}^2 - Y^2 \right) \quad \text{or} \quad \frac{dY}{dx} = -\frac{\lambda}{x^2} \left( Y^2 - Y_{\text{EQ}}^2 \right)
\]

- Riccati equation with $\lambda = m_X^3 \langle v \sigma \rangle / H_m$

---

Thomas Gajdosik – Concepts of Modern Cosmology 23.04.2012 21
7. General Relativity — dark matter

dark matter balance

- for estimating $\lambda = m_X^3 \langle v \sigma \rangle / H_m$ we need
  - the mass $m_X$
  - the cross section $\sigma_{X+X \rightarrow Y+Z}$
  - the Hubble parameter $H_m$ at the mass scale $m_X$

- for $X$ from Supersymmetry (SUSY) we know the cross section $\sigma$
  - we get limits on the mass $m_X$
  - $v$ and $H_m$ are given by the Einstein-Boltzmann equations
  $\Rightarrow$ for the lightest supersymmetric particle (LSP) $\neq$ gravitino

$$\Omega_X \sim 0.3 \left( \frac{x_f}{10} \right) \left( \frac{g_*(m_X)}{100} \right)^{1/2} \frac{10^{-39} \text{cm}^2}{\langle v \sigma \rangle}$$

- the gravitino couples as $\frac{E_X^2}{M_P^2} \sim \frac{(10^3 \text{GeV})^2}{(10^{19} \text{GeV})^2} \sim 10^{-32} \ll \alpha_{\text{em}}$

  $\Rightarrow$ it is only at the Plank epoch in thermal equilibrium

* all estimates for $\Omega_X$ have to be reassessed