5. General Relativity — physical cosmology

we have learned so far:

- SR as the background to non-gravitational physics
  - Lorentz transformations
  - their application to astronomical observations:
    - relativistic Doppler effect $\sim$ astronomical redshift
    - relativistic "beaming"

- GR as the covariant inclusion of gravity into SR
  - mathematical background of Riemannian geometry
    - curvature as an intrisic property of a manifold
  - validity of a local Lorentz frame:

    $\Rightarrow$ physics on Earth is SR physics

- vacuum solutions for GR ... and their properties
- Friedmann-Lemaître models for GR ... and their properties
  - relation between scale parameter, time, distance, and energy content
- but no observations ... yet

what can we observe?

- light ... and only very recently also cosmic electrons, protons, and neutrinos
  - electromagnetic radiation
    - in different wavelengths: 8 pm (40 EHz) to 670 μm (450GHz) up to 2m (144kHz)
    - in different intensities: apparent magnitude from -27 (sun) to +32 (limit of HST)
    - from different directions: "the whole sky" (4π spherical surface)

⇒ differentiating the light into different "identified" objects
  - solar system objects: sun, planetes, and moons
  - stars
    - in our own galaxy (the Milky Way)
    - supernovae: also in other galaxies ... "standard candles"
  - galaxies: different types of galaxies
  - nebulae (clouds): interstellar dust
  - cosmic (microwave) background radiation (CMB) or (CBR)

what do we conclude?

- understanding the solar sytem observations
  - ⇒ Keplers laws and Newtonian gravity ... Copernicus principle
- understanding the spectral lines of hydrogen
  - – measuring the redshift of the light + SR
  - ⇒ velocity of the emitter
    - * Cepheids (variable stars) ⇒ Hubbles law
    - * parts of galaxies ⇒ rotation curves
    - * supernovae Ia ⇒ modern measurement of Hubbles law
- measuring the CMB
  - – perfect blackbody radiation, scale invariant, isotropic
    - * with a dipole part: movement against the isotropic background
    - * we calculated the effect in SR2 ... but more later
- galaxy surveys ⇒ universe seems *roughly* isotropic

⇒ confidence in homogeneity (from theory) and isotropy (observational)

⇒ confidence in Friedmann-Lemaître models ... as a first approximation

Cosmological units

- the basic cosmological unit is the Hubble parameter

$$H_0 \sim 69.32 \pm 0.80 \frac{\text{km/s}}{\text{Mpc}} \sim 2.24 \times 10^{-18}/\text{s}$$ measured 2012 by WMAP

  - derived from that is the Hubble time

    $$t_H = 1/H_0 \sim 4.46 \times 10^{17}\text{s} = 14.125 \times 10^{9}\text{yr} = 14.125\,\text{Gyr}$$

  - derived from that is the Hubble length

    $$\ell_H = ct_H \sim 1.336 \times 10^{26}\text{m} = 4331\,\text{Mpc}$$

- using Newton's gravitational constant $G_N = 6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$
  - one defines the Planck mass $M_P = \sqrt{\frac{\hbar c}{G_N}}$ or the Planck energy

    $$E_P = M_P c^2 = \sqrt{\hbar c^5/G_N} = 1.22 \times 10^{19}\text{GeV}$$

  - the Planck length

    $$\ell_P = \hbar c/E_P = \sqrt{\hbar G_N/c^3} = 1.62 \times 10^{-35}\,\text{m}$$

  - and the Planck time

    $$t_P = \ell_P/c = \hbar/E_P = \sqrt{\hbar G_N/c^5} = 5.39 \times 10^{-44}\,\text{s}$$

mass density

- the critical density

$$\rho_c = \frac{3H_0^2}{8\pi G_N} \sim 9.7 \times 10^{-27}\,\frac{\text{kg}}{\text{m}^3} \sim 5.45\frac{\text{GeV}/c^2}{\text{m}^3} \sim 5.8\,\text{protons}/\text{m}^3$$

- the density of luminous baryonic matter (i.e. stars) is estimated
  - by the product of the luminosity density $\rho_L \approx 2 \times 10^8 L_\odot/\text{Mpc}^3$
    - estimated from galaxy counts
  - and the mass over luminosity ratio $M/L \approx 4M_\odot/L_\odot$
    - averaged over different samples of stars and galaxies

$$\rho_{\text{lum}} \sim 2 \times 10^8 L_\odot/\text{Mpc}^3 \times 4M_\odot/L_\odot \sim 8 \times 10^8 \cdot 1.99 \times 10^{30}\text{kg}/(30.857 \times 10^{15}\text{m})^3$$

$$\sim 5.4 \times 10^{-29}\frac{\text{kg}}{\text{m}^3} \sim 0.03\frac{\text{GeV}/c^2}{\text{m}^3} \sim 0.032\,\text{protons}/\text{m}^3$$

$\Rightarrow$ that gives a density $\Omega_{\text{lum}} \sim 0.0056$

- there is also interstellar nonluminous gas
  - mainly hydrogen and helium
  - hydrogen absoption lines were seen in Quasar spectra, indicating $\Omega_{\text{gas}} \sim 0.04$

  $\Rightarrow$ that gives a baryonic density $\Omega_{\text{B}} = \Omega_{\text{lum}} + \Omega_{\text{gas}} \sim 0.0056 + 0.04 = 0.0456$
  - a consistent estimate comes from Big Bang nucleosynthesis: $\Omega_{\text{B}} \sim 0.043$

radiation

- obviously from stars
- also from the cosmic background radiation (CBR)
  - a blackbody radiation with a temperature of $T = 2.725$ K
  - counting the number density of photons we get

$$dn = \frac{g}{h^3c^3} \cdot \frac{d^3p}{e^{E/k_BT} - 1} = \frac{4\pi g}{(2\pi)^3\hbar^3c^3} \cdot \frac{E^2\,dE}{e^{E/k_BT} - 1}$$

    * g counts the possible states: for the photon $g = 2$ polarization states
  - using $\int_0^\infty \frac{x^2\mathrm{d}x}{e^x-1} = 2\,\zeta(3) \sim 2.40411$ we can integrate to get

$$n_\gamma = \frac{\zeta(3)}{\pi^2} g \left(\frac{k_BT}{\hbar c}\right)^3 \sim 4.1 \times 10^8/m^3$$

- the energy density of these photons is

$$d\rho_\gamma = \frac{g}{h^3c^3} \cdot \frac{Ed^3p}{e^{E/k_BT} - 1} = \frac{4\pi g}{h^3c^3} \cdot \frac{E^3\,dE}{e^{E/k_BT} - 1} \quad \text{or} \quad \rho_\gamma = \frac{\hbar c\, g}{2\pi^2} \frac{\pi^4}{15} \left(\frac{k_BT}{\hbar c}\right)^4 \sim 0.264 \frac{\text{MeV}}{\text{m}^3}$$

$\Rightarrow$ $\Omega_\gamma = \rho_\gamma/\rho_c \sim \frac{0.264\text{MeV}}{5.45\text{GeV}} \sim 4.85 \times 10^{-5}$

  - three orders of magnitude smaller than Ω_B

radiation

- we know from the conservation of the stress energy tensor
  - ρ_γ scales with the scale factor a as $\rho_\gamma \propto a^{-4} \propto (1+z)^4$
- we saw from the Stefan-Boltzmann law $\rho_\gamma \propto T^4$
  - derived before by integrating the energy density according to Bose-Einstein statistics

⇒ the temperature T scales like the inverse scale factor $T \propto a^{-1}$
  - for a radiation dominated universe

- Thermodynamics tells
  - interacting systems try to reach thermal equilibrium
  - baryonic matter and radiation can interact

  ⇒ $T_{\text{radiation}} \approx T_{\text{baryonic matter}}$ in equilibrium

  * we can call this radiation temperature the temperature of the universe

- when the scale factor shrinks (back in time) the temperature grows

  ⇒ a Hot Big Bang

cosmological baryonic matter today is described as dust

- no interaction between its ”molecules”
  - these molecules are
    - hydrogen atoms of interstellar clouds
    - stars
    - galaxies, galaxy clusters, . . .
- their interaction to radiation is very weak, too
  - stars and galaxies emit light
    - but the absorbtion does not change their state
  - interstellar clouds (nebulae) emit/rescatter the light they absorb
    - ⇒ there it makes more sense to talk about their temperature

⇒ description of cosmological baryonic matter as a perfect fluid without pressure is justified

- only radiation interacts and has pressure in this picture

inhomogeneities in the dust

- Einstein's equations are local

  ⇒ they describe parts of the universe, too

  - ∗ for instance a gas cloud . . .

- "handwaving" description:
  - outer atoms are attracted to the center of the cloud
  - they gain kinetic energy from their fall in the gravitational potential
  - they scatter ⇒ energy distributes ⇒ gas heats up ⇒ gas pressure

  ⇒ thermodynamic description of the gas cloud when pressure stops the collaps

  ⇒ Einstein–Boltzmann equations for the exact description

- what happens to a gas cloud?
  - Newton: denser regions experience stronger gravitational attraction
    - ∗ a dilute gas of hydrogen has a low density ⇒ weak gravitational field
    - ∗ being initially at rest ⇒ low velocity of the atoms

    ⇒ Newtonian limit applicable

  ⇒ inhomogeneities grow through gravitation: gravitational collaps

description of the gas cloud

- Newtonian limit allowed (initially) $\Rightarrow$ linearized Einstein equations
  - linearizing the metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$
  - assuming homogeneity and isotropy
    * they are only slightly perturbed

  $\Rightarrow$ use Robertson-Walker line element (for the locally flat background)

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \approx dt^2 - a^2(t)\eta_{jk}\, dx^j\, dx^k + \mathcal{O}(h)$$

    * $\eta_{jk} = \delta_{jk}$ describes the Euclidean metric of flat space

  - a multipole expansion of the metric perturbations $h_{\mu\nu}$ along the $\hat{z}$-axis gives
    * scalar perturbations: $h_{00} = 2\Psi$ and $h_{jk} = \eta_{jk}\, 2\Phi\, a^2$
    * vector perturbations: only h_{13} and h_{23} are non-zero
    * tensor perturbations: only $h_{11} = -h_{22} =: h_+$ and $h_{12} =: h_\times$ are non-zero

- a density contrast δ on top of a background density ρ_{bg}

$$\rho(t, \vec{x}) = \rho_{\text{bg}}(t, \vec{x})[1 + \delta(t, \vec{x})]$$

stress energy of the gas cloud

- the stress energy tensor in comoving coordinates

$$T_{\mu\nu} = \rho\, U_\mu U_\nu - \mathbf{p}(g_{\mu\nu} - U_\mu U_\nu) = g_{\mu\lambda}[\rho\, U^\lambda U_\nu - \mathbf{p}(\delta^\lambda_\nu - U^\lambda U_\nu)]$$

  – was composed from its components

$$T^0{}_0 = \rho = \rho_{\text{bg}}(1+\delta) \quad \text{and} \quad T^j{}_k = -\mathbf{p}\,\delta^j_k$$

- using the equation of state $\mathbf{p} = w\rho$

  ⇒ we only need the density and not anymore the pressure

conservation of the stress energy $\nabla^\mu T_{\mu\nu} = 0$

- requires the Christoffel symbols in terms of the metric perturbations $h_{\mu\nu}$
- evaluating the conservation law to zero order in δ and h

  ⇒ the normal evolution of the background density $\rho_{\text{bg}} \propto a^{-3(1+w)}$

  = consistency condition

5. General Relativity — structure formation

conservation of the stress energy $\nabla^\mu T_{\mu\nu} = 0$

- evaluating the conservation law to first order in δ and h
  - a relation between derivatives of the parameters

    $\Rightarrow$ the perturbation expansions in δ and h are linked:

$$\mathcal{O}(\delta^n) = \mathcal{O}(h^n)$$

  = only a single perturbation expansion

linearized Einstein equations

- solving the perturbation series in δ and h
  - zero order $\Rightarrow$ normal Friedmann equations with ρ_{bg}
  - first order $\Rightarrow$ evolution of the density contrast δ

$\Rightarrow$ seeds for structure formation
  - cosmic Walls
  - cosmic Strings
  - point-like (spherically symmetric)

calculating the time for a gas cloud to collapse

- taking a spherical cloud of constant mass M and initial radius r_0
- let the outer shell of thickness dr contract under the gravity of the cloud
- its mass $m = 4\pi\rho r^2\, dr$ and its Newtonian gravitational potential $V(r) = -\frac{GMm}{r}$
- falling from r_0 to r converts the potential difference $\Delta V = -\frac{GMm}{r_0} + \frac{GMm}{r}$ into kinetic energy $\Delta E = E(r) - E(r_0) = \frac{m}{2}[v^2 - v_0^2]$ (Newtonian limit)
- the velocity $v(r) = \frac{dr}{dt} = [2GM(r^{-1} - r_0^{-1})]^{1/2}$ gives the free falling time

$$t_{ff} = \int_0^{t_{ff}} \frac{dr}{dr/dt} = \int_{r_0}^{r} \frac{dr'}{[2GM(1/r' - 1/r_0)]^{1/2}} \xrightarrow{r\to 0} \frac{\pi}{2}\left[\frac{r_0^3}{2GM}\right]^{1/2} = \left[\frac{3\pi}{32G\rho}\right]^{1/2}$$

  – taking the density in terms of the critical density $\rho = \Omega\rho_c$

$$t_{ff} = \sqrt{3\pi}\left[32G\,\Omega\frac{3H_0^2}{8\pi G}\right]^{-1/2} = \frac{\pi}{2H_0\sqrt{\Omega}} = \frac{\pi\, t_H}{2\sqrt{\Omega}}$$

- before collapsing into a point, the thermodynamic pressure might stop the collaps
  – igniting of a star . . . the first stars formed $\sim t_H/30$
  $\Rightarrow \frac{\pi\, t_H}{2\sqrt{\Omega}} \sim \frac{t_H}{30}$ or $(15\pi)^2 \sim \Omega = \Omega_{\text{bg}}[1+\delta]$ or $1+\delta \sim \frac{(15\pi)^2}{\Omega_{\text{bg}}}$
  – using the baryonic density $\Omega_{\text{B}} = 0.0456 \quad\Rightarrow\quad 1+\delta \sim 5 * 10^5 \gg 1$
  $\Rightarrow$ non-linear effects, linearized equations not enough $\Rightarrow$ full simulation
- we need additional matter for structure formation . . . Dark Matter

Dark Matter

- "invisible" in the sky
  - like planets, neutron stars, ...
- helps with the rotation curves of galaxies

particle dark matter: has to be "invisible", too

- neutral, only minimally coupled to light
  - ⇒ decouples earlier from the radiation
  - helps also in the initial phases of structure formation
- can not interact with nuclear of strong interactions
  - otherwise we would have seen it bound to atomic nuclei
  - ⇒ anomalous isotopes
- massive: with low kinetic energy today
  - otherwise it would wash out the density contrast

⇒ **WIMPs** (weakly interacting massive particles)

- weak interactions, like very heavy neutrinos: neutralinos, sneutrinos from SUSY
- or only gravitationally interacting ("GIMPs"): gravitinos

looking back ...assuming the best fit from $d_L(z)$ and other measurments

$$\Omega_\gamma \sim 4.85 \times 10^{-5} \qquad \Omega_m \sim 0.2736 \qquad \Omega_k \sim 0 \qquad \Omega_\Lambda \sim 0.726$$

$\Rightarrow$ we can reconstruct a timeline in z:

z	time		T[K]	description
0.	13.72	Gyr	2.725	now
0.38	9.56	Gyr	3.76	equality of matter density and cosmological constant
0.6	8.	Gyr	4.36	formation of the solar system
1.	6.	Gyr	5.45	formation of superclusters, walls, and voids
1.3	4.9	Gyr	6.27	formation of the thin disk of the Milky Way
10.9	.43	Gyr	32.4	reionization: first stars are ionizing the interstellar gas again
940.	.5	Myr	2564.	equality of baryonic matter and radiation
1040.	445.	kyr	2837.	Baryon decoupling and drag epoch
1100.	406.	kyr	3000.	Photon decoupling and last scattering (CMB)
1370.	283.	kyr	3736.	Recombination of e^- and ions: roughly neutral universe
3200.	67.	kyr	8723.	Matter-radiation equality, including neutrinos
5600.	25.	kyr	15×10^3	radiation dominance: $\rho_\gamma > \rho_m$
8.5×10^8	32.	h	$\sim 10^7$	Photon reheating: e^+-e^- recombination
2.6×10^9	10.	min	1.2×10^9	Big Bang nucleosynthesis

starting from the Big Bang

- we can suggest a timeline in t:

time until	T[GeV]	description
10^{-43} s	10^{19} GeV	Planck epoch: t_P, E_P
10^{-36} s	10^{16} GeV	Grand unification epoch: all the forces of the SM are unified
10^{-32} s (?)	10^{14} GeV	end of inflation, Baryogenesis, Supersymmetry breaking
10^{-12} s	10^{3} GeV	electroweak symmetry breaking
10^{-6} s	1 GeV	quark gluon plasma
0.8 s	1.1 MeV	neutrino decoupling
1 s	1 MeV	hadronic phase
10 s	0.3 MeV	leptonic phase: photon reheating though pair annihilation
20 min	25 keV	nucleosynthesis
70 kyrs	2 eV	Matter domination, start of structure formation
380 kyrs	0.7 eV	recombination: matter becomes transparent to light $\Rightarrow$ CMB pattern

- description of these early times needs particle physics
  $\Rightarrow$ astro particle physics ... statistical description $\Rightarrow$ Boltzmann equations