2. General Relativity — the metric

mathematical definition:

- a metric in M is a symmetric, bilinear, non-degenerate function

$$g_P : T_P \times T_P \to \mathbf{R}$$

  - the metric acts on (contravariant) vectors (tensor indices)
  - symmetric means $g(X,Y) = g(Y,X) \in \mathbf{R}$ for $X,Y \in T_P$
  - bilinear means for $X,Y,Z \in T_P$ and $a,b,c \in \mathbf{R}$

$$g(aX + bY, cZ) = ac\, g(X,Z) + bc\, g(Y,Z) \in \mathbf{R}$$

  - non-degenerate: $g(X,Y) = 0$ for all $Y \in M$ only if $X = 0$
  - acting on the basis vectors $\vec{e}_{(\mu)}$ gives the metric tensor

$$g_{\mu\nu}(P) = g(\vec{e}_{(\mu)}, \vec{e}_{(\nu)})\Big|_P$$

- the metric allows length and angle measurements

$$\|X\| := \sqrt{|g(X,X)|} \qquad \text{and} \qquad \cos\varphi = \frac{g(X,Y)}{\|X\| * \|Y\|}$$

examples for the metric

- in Euclidean space the metric is the normal dot-product:

$$\vec{a} \cdot \vec{b} = g_{jk} a^j b^k = \delta_{jk} a^j b^k \qquad \text{so} \quad g_{j=k} = 1 \quad \text{and} \quad g_{j \neq k} = 0$$

  – we can use this metric of the embedding to get the induced metric in S^2

$$g_{jk} = g(\vec{e}_{(j)}, \vec{e}_{(k)}) = \vec{e}_{(j)} \cdot \vec{e}_{(k)} \qquad \text{so} \quad g_{11} = s^2_{\vartheta_0} \qquad g_{12} = g_{21} = 0 \qquad g_{22} = 1$$

$\Rightarrow$ as we can see, the metric tensor depends on the position

- in Minkovsky space we had $g_{00} = 1$, $g_{ii} = -1$, and $g_{\mu \neq \nu} = 0$
  – we can generalize the line element

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = g_{\mu\nu} \Delta x^\mu \otimes \Delta x^\nu$$

  – the "differentials" Δx here are **not** multiplied with their "natural" wedge product

- one can define the inverse metric $g^{\mu\nu}$ by

$$g^{\mu\nu} g_{\mu\rho} = \delta^\nu_\rho$$

  – the inverse metric is the metric of the cotangent space

connection and the metric

- the metric gives angles
  - ⇒ so we can relate angles between vector fields all over the manifold
- but we have not specified the vectorfield, that defines parallel transport
  - that is done with the connection / gauge field
  - the metric has to be compatible with this choice
- as seen with the Lie bracket, parallel transport has to do with derivation
  - the Lie derivative was a map $\mathcal{L}(T_P) : T_P \to T_P$
    - leaving the differentiated object in its own representation
  - the directional derivative was a map $D(T_P) : \mathbf{R} \to T_P$
  - ⇒ this is, what we want
    - but it does not connect the different points in M
  - ⇒ including an affine connection Γ to make a covariant derivative ∇
    - like in gauge theories: $\nabla = \partial + \Gamma = d + \omega$

definition for a covariant derivative:

- the covariant derivative ∇_U in the direction of U
  - should map tensors to tensors
  - be linear in the direction: $\nabla_{fU+gV}\,T = f\nabla_U\,T + g\nabla_V\,T$
  - be linear in the argument: $\nabla_U\,(T+S) = \nabla_U\,T + \nabla_U\,S$
  - follow the Leibniz rule: $\nabla_U\,(T\otimes S) = (\nabla_U\,T)\otimes S + T\otimes(\nabla_U\,S)$
  - commute with contractions: $\nabla_U\,\delta^\lambda_\rho = 0$
    * this means in index notation $\nabla_\mu(T^\lambda{}_{\lambda\nu}) = \nabla_\mu(\delta^\lambda_\rho T^\rho{}_{\lambda\nu}) = \delta^\lambda_\rho\nabla_\mu(T^\rho{}_{\lambda\nu}) = (\nabla T)^\lambda{}_{\lambda\nu;\mu}$
  - be the partial derivative on scalars: $\nabla_U\,\phi = U(\phi) = (U)^\mu\partial_\mu\phi$

definition of the Levi Civita connection:

- the covariant derivative ∇_U in the direction of U
  - is metric compatible: $\nabla_U\,g(X,Y) = 0$
  - is torsion free: $(\nabla_X\,Y - \nabla_Y\,X) = [X,Y]$
    * this is also easier explained in index notation ...

using the natural basis

- the covariant derivative ∇_μ goes into the direction of $\partial_{(\mu)}$
- the covariant derivative in the direction of U is $\nabla_U = U^\mu \nabla_\mu$
  - linearity in the direction is obvious from $U \in T_P(M)$

writing the Levi Civita connection as:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\rho} A^\rho \qquad \text{and} \qquad \nabla_\mu a_\nu = \partial_\mu a_\nu + \Gamma'^\rho_{\mu\nu} a_\rho$$

- commutation with contractions means:
  - with $B = A^\mu a_\mu = \delta^\lambda_\rho A^\rho a_\lambda$ a scalar:

$$\begin{aligned} \nabla_\mu B &= \delta^\lambda_\rho [(\partial_\mu A^\rho + \Gamma^\rho_{\mu\nu} A^\nu) a_\lambda + A^\rho (\partial_\mu a_\lambda + \Gamma'^\nu_{\mu\lambda} a_\nu)] \\ &= (\partial_\mu A^\rho) a_\rho + A^\rho (\partial_\mu a_\rho) + [\Gamma^\rho_{\mu\nu} A^\nu a_\rho + A^\lambda \Gamma'^\nu_{\mu\lambda} a_\nu] \\ &= \partial_\mu (A^\nu a_\nu) + A^\nu a_\rho [\Gamma^\rho_{\mu\nu} + \Gamma'^\rho_{\mu\nu}] \end{aligned}$$

$\Rightarrow$ $\Gamma'^\rho_{\mu\nu} = -\Gamma^\rho_{\mu\nu}$

  * covariant indices transform opposite from contravariant ones

writing the Levi Civita connection as:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\rho} A^\rho \qquad \text{and} \qquad \nabla_\mu a_\nu = \partial_\mu a_\nu - \Gamma^\rho_{\mu\nu} a_\rho$$

- torsion free means:

$$\begin{aligned} 0 &= X^\mu \nabla_\mu Y^\nu - Y^\mu \nabla_\mu X^\nu - [X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu] \\ &= X^\mu \Gamma^\nu_{\mu\rho} Y^\rho - Y^\mu \Gamma^\nu_{\mu\rho} X^\rho = X^\mu Y^\rho (\Gamma^\nu_{\mu\rho} - \Gamma^\nu_{\rho\mu}) \end{aligned}$$

⇒ the Christoffel symbols $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ are symmetric

- metric compatible means:

$$0 = \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\lambda\mu}$$

  - adding $\nabla_\mu g_{\nu\rho} + \nabla_\nu g_{\mu\rho} - \nabla_\rho g_{\mu\nu}$ we get

$$\begin{aligned} 0 &= \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\lambda\nu} + \partial_\nu g_{\mu\rho} - \Gamma^\lambda_{\nu\mu} g_{\lambda\rho} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \partial_\rho g_{\mu\nu} + \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} + \Gamma^\lambda_{\rho\nu} g_{\lambda\mu} \\ &= \partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} - 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} \end{aligned}$$

⇒ the Christoffel symbols can be calculated from the metric:

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

The difference between a Riemannian manifold and flat space

- parallel transport of a vector V along a curve $C(t)$ is defined by

$$\nabla_{\frac{d}{dt}C(t)} V = 0$$

- taking the curve as a Levi Civita parallelogramoid
  - along the vectors A and B

$$P_{00} \xrightarrow{A(\Delta s)} P_{10} \xrightarrow{B'(\Delta t)} P_{11} \xrightarrow{A'(-\Delta s)} P_{01} \xrightarrow{B(-\Delta t)} P_{00}$$

    * with B' parallel transported along A, so $\nabla_A B = 0$, and $B' = B$
  - we can compare the vector at P_{11} between the two different paths

$$P_{00} \xrightarrow{A(\Delta s)} P_{10} \xrightarrow{B(\Delta t)} P_{11} \qquad \text{and} \qquad P_{00} \xrightarrow{B(\Delta t)} P_{01} \xrightarrow{A(\Delta s)} P_{11}$$

  - and the difference of the parallel transported vector at P_{11} is

$$\nabla_B \nabla_A V - \nabla_A \nabla_B V$$

The difference between a Riemannian manifold and flat space

- evaluating the difference in the natural basis ∂_μ we get

$$\begin{aligned} \nabla_B\nabla_A V - \nabla_A\nabla_B V &= B^\sigma\nabla_\sigma A^\rho\nabla_\rho V^\mu\partial_\mu - A^\rho\nabla_\rho B^\sigma\nabla_\sigma V^\mu\partial_\mu \\ &= A^\rho B^\sigma(\nabla_\sigma\nabla_\rho - \nabla_\rho\nabla_\sigma)V^\mu\partial_\mu \end{aligned}$$

  - due to the condition $\nabla_A B = \nabla_B A = 0$

- using the affine connection we get $(\nabla_\sigma\nabla_\rho - \nabla_\rho\nabla_\sigma)V^\mu$

$$\begin{aligned} &= \nabla_\sigma(\partial_\rho V^\mu + \Gamma^\mu_{\rho\nu}V^\nu) - \nabla_\rho(\partial_\sigma V^\mu + \Gamma^\mu_{\sigma\nu}V^\nu) \\ &= \partial_\sigma(\partial_\rho V^\mu + \Gamma^\mu_{\rho\nu}V^\nu) - \Gamma^\lambda_{\sigma\rho}(\partial_\lambda V^\mu + \Gamma^\mu_{\lambda\nu}V^\nu) + \Gamma^\mu_{\sigma\lambda}(\partial_\rho V^\lambda + \Gamma^\lambda_{\rho\nu}V^\nu) \\ &- \partial_\rho(\partial_\sigma V^\mu + \Gamma^\mu_{\sigma\nu}V^\nu) + \Gamma^\lambda_{\rho\sigma}(\partial_\lambda V^\mu + \Gamma^\mu_{\lambda\nu}V^\nu) - \Gamma^\mu_{\rho\lambda}(\partial_\sigma V^\lambda + \Gamma^\lambda_{\sigma\nu}V^\nu) \\ &= (\partial_\sigma\Gamma^\mu_{\rho\nu} - \partial_\rho\Gamma^\mu_{\sigma\nu} - \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu} + \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\rho\nu})V^\nu \\ &\quad -(\Gamma^\lambda_{\sigma\rho} - \Gamma^\lambda_{\rho\sigma})(\partial_\lambda V^\mu + \Gamma^\mu_{\lambda\nu}V^\nu) \\ &=: R^\mu{}_{\nu\sigma\rho}V^\nu - T^\nu_{\sigma\rho}(\nabla_\nu V^\mu) \end{aligned}$$

⇒ the Riemann curvature tensor $R^\mu{}_{\nu\sigma\rho}$ and the Torsion tensor $T^\nu_{\sigma\rho}$

  - $T^\nu_{\rho\sigma}$ vanishes for the Christoffel symbols $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$

The difference between a Riemannian manifold and flat space

- in Euclidean (or Minkovsky) flat space one can use
  - Cartesian coordinates, where the metric is constant
    - $\Rightarrow$ the connection and the Riemann tensor vanish identically
  - curvlinear coordinates, where the metric is not constant
    - $\Rightarrow$ the connection is non zero
      - $*$ but the Riemann tensor still vanishes identically

- example: spherical coordinates in $\mathbf{R}^3$
  - with unit vectors $e_{(r)} = (s_\vartheta c_\varphi, s_\vartheta s_\varphi, c_\vartheta)$, $e_{(\vartheta)} = (c_\vartheta c_\varphi, c_\vartheta s_\varphi, -s_\vartheta)$, and $e_{(\varphi)} = (-s_\varphi, c_\varphi, 0)$
  - the metric is $ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2$
    - $*$ with non-zero coefficients $g_{rr} = 1$, $g_{\vartheta\vartheta} = r^2$, and $g_{\varphi\varphi} = r^2 \sin^2\vartheta$
  - the Christoffel symbols are

$$
\begin{array}{rcl}
\Gamma^r_{\mu\nu} &=& \frac{1}{2} g^{rr}(\partial_\mu g_{\nu r} + \partial_\nu g_{\mu r} - \partial_r g_{\mu\nu}) \quad \Rightarrow \quad \Gamma^r_{\vartheta\vartheta} = -r \quad \Gamma^r_{\varphi\varphi} = -r\sin^2\vartheta \\
\Gamma^\vartheta_{\mu\nu} &=& \frac{1}{2} g^{\vartheta\vartheta}(\partial_\mu g_{\nu\vartheta} + \partial_\nu g_{\mu\vartheta} - \partial_\vartheta g_{\mu\nu}) \quad \Rightarrow \quad \Gamma^\vartheta_{r\vartheta} = r^{-1} \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \\
\Gamma^\varphi_{\mu\nu} &=& \frac{1}{2} g^{\varphi\varphi}(\partial_\mu g_{\nu\varphi} + \partial_\nu g_{\mu\varphi} - \partial_\varphi g_{\mu\nu}) \quad \Rightarrow \quad \Gamma^\varphi_{r\varphi} = r^{-1} \quad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta
\end{array}
$$

The difference between a Riemannian manifold and flat space

- example: spherical coordinates in $\mathbf{R}^3$
  - the Riemann tensor $R^\mu{}_{\nu\sigma\rho} = (\partial_\sigma \Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu}) - (\rho \leftrightarrow \sigma)$
  - in components $R^\mu{}_{\nu\sigma\rho} = (\partial_\sigma \Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\rho r}\Gamma^r_{\sigma\nu} - \Gamma^\mu_{\rho\vartheta}\Gamma^\vartheta_{\sigma\nu} - \Gamma^\mu_{\rho\varphi}\Gamma^\varphi_{\sigma\nu}) - (\rho \leftrightarrow \sigma)$

$$\begin{aligned}
R^r_{r\sigma\rho} &= (\partial_\sigma \Gamma^r_{\rho r} - \Gamma^r_{\rho r}\Gamma^r_{\sigma r} - \Gamma^r_{\rho\vartheta}\Gamma^\vartheta_{\sigma r} - \Gamma^r_{\rho\varphi}\Gamma^\varphi_{\sigma r}) - (\rho \leftrightarrow \sigma)\\
&= (\partial_\sigma 0 - 00 - (-r)\delta^\vartheta_\rho (r^{-1})\delta^\vartheta_\sigma - (-r\sin^2\vartheta)\delta^\varphi_\rho (r^{-1})\delta^\varphi_\sigma) - (\rho \leftrightarrow \sigma)\\
&= (\delta^\vartheta_\rho\delta^\vartheta_\sigma + \sin^2\vartheta\, \delta^\varphi_\rho\delta^\varphi_\sigma) - (\rho \leftrightarrow \sigma) = 0
\end{aligned}$$

$$\begin{aligned}
R^r_{\vartheta\sigma\rho} &= (\partial_\sigma \Gamma^r_{\rho\vartheta} - 0\Gamma^r_{\sigma\vartheta} - \Gamma^r_{\rho\vartheta}\Gamma^\vartheta_{\sigma\vartheta} - \Gamma^r_{\rho\varphi}\Gamma^\varphi_{\sigma\vartheta}) - (\rho \leftrightarrow \sigma)\\
&= (\partial_\sigma(-r)\delta^\vartheta_\rho - (-r)\delta^\vartheta_\rho (r^{-1})\delta^r_\sigma - (-r\sin^2\vartheta)\delta^\varphi_\rho(\cot\vartheta)\delta^\varphi_\sigma) - (\rho \leftrightarrow \sigma)\\
&= (-\delta^r_\sigma\delta^\vartheta_\rho + \delta^\vartheta_\rho\delta^r_\sigma) - (\rho \leftrightarrow \sigma) = 0
\end{aligned}$$

$$\begin{aligned}
R^r_{\varphi\sigma\rho} &= (\partial_\sigma \Gamma^r_{\rho\varphi} - 0\Gamma^r_{\sigma\varphi} - \Gamma^r_{\rho\vartheta}\Gamma^\vartheta_{\sigma\varphi} - \Gamma^r_{\rho\varphi}\Gamma^\varphi_{\sigma\varphi}) - (\rho \leftrightarrow \sigma)\\
&= (\partial_\sigma(-rs^2_\vartheta)\delta^\varphi_\rho - (-r)\delta^\vartheta_\rho(-s_\vartheta c_\vartheta)\delta^\varphi_\sigma - (-rs^2_\vartheta)\delta^\varphi_\rho[(r^{-1})\delta^r_\sigma + (c_\vartheta/s_\vartheta)\delta^\vartheta_\sigma]) - (\rho \leftrightarrow \sigma)\\
&= (-[\delta^r_\sigma s^2_\vartheta + r2s_\vartheta c_\vartheta\delta^\vartheta_\sigma]\delta^\varphi_\rho - rs_\vartheta c_\vartheta\delta^\vartheta_\rho\delta^\varphi_\sigma + s^2_\vartheta\delta^\varphi_\rho\delta^r_\sigma + rs_\vartheta c_\vartheta\delta^\varphi_\rho\delta^\vartheta_\sigma) - (\rho \leftrightarrow \sigma)\\
&= -rs_\vartheta c_\vartheta(2\delta^\varphi_\rho\delta^\vartheta_\sigma + \delta^\varphi_\sigma\delta^\vartheta_\rho - \delta^\varphi_\rho\delta^\vartheta_\sigma) - (\rho \leftrightarrow \sigma) = 0
\end{aligned}$$

  - and similar for $R^\vartheta_{\nu\sigma\rho}$ and $R^\varphi_{\nu\sigma\rho}$

⇒ $\mathbf{R}^3$ is flat also in spherical coordinates

The difference between a Riemannian manifold and flat space

- example: S^2 with coordinates $(x^1 = \varphi, x^2 = \vartheta)$ and basis vectors $e_{(1)}$, $e_{(2)}$
  - embedded in $\mathbf{R}^3$ with $(x, y, z) = r(s_\vartheta c_\varphi, s_\vartheta s_\varphi, c_\vartheta)$ and r constant
  - the metric is $ds^2 = (dx^1)^2 + (dx^2)^2 = r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2$
    - with non-zero coefficients $g_{\vartheta\vartheta} = r^2$ and $g_{\varphi\varphi} = r^2 \sin^2\vartheta$
  - the Christoffel symbols are

$$\begin{aligned}
\Gamma^\vartheta_{\mu\nu} &= \tfrac{1}{2} g^{\vartheta\vartheta}(\partial_\mu g_{\nu\vartheta} + \partial_\nu g_{\mu\vartheta} - \partial_\vartheta g_{\mu\nu}) \quad \Rightarrow \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \\
\Gamma^\varphi_{\mu\nu} &= \tfrac{1}{2} g^{\varphi\varphi}(\partial_\mu g_{\nu\varphi} + \partial_\nu g_{\mu\varphi} - \partial_\varphi g_{\mu\nu}) \quad \Rightarrow \quad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta
\end{aligned}$$

  - and the Riemann tensor $R^\mu{}_{\nu\sigma\rho} = (\partial_\sigma \Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\sigma\nu}) - (\rho \leftrightarrow \sigma)$
  - in components $R^\mu{}_{\nu\sigma\rho} = (\partial_\sigma \Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\rho\vartheta}\Gamma^\vartheta_{\sigma\nu} - \Gamma^\mu_{\rho\varphi}\Gamma^\varphi_{\sigma\nu}) - (\rho \leftrightarrow \sigma)$

$$\begin{aligned}
R^\vartheta_{\vartheta\sigma\rho} &= (\partial_\sigma \Gamma^\vartheta_{\rho\vartheta} - \Gamma^\vartheta_{\rho\vartheta}\Gamma^\vartheta_{\sigma\vartheta} - \Gamma^\vartheta_{\rho\varphi}\Gamma^\varphi_{\sigma\vartheta}) - (\rho \leftrightarrow \sigma) = (-(-\sin\vartheta\cos\vartheta)\delta^\varphi_\rho(\cot\vartheta)\delta^\varphi_\sigma) - (\rho \leftrightarrow \sigma) = 0 \\
R^\vartheta_{\varphi\sigma\rho} &= (\partial_\sigma \Gamma^\vartheta_{\rho\varphi} - \Gamma^\vartheta_{\rho\vartheta}\Gamma^\vartheta_{\sigma\varphi} - \Gamma^\vartheta_{\rho\varphi}\Gamma^\varphi_{\sigma\varphi}) - (\rho \leftrightarrow \sigma) = (\partial_\sigma(-\sin\vartheta\cos\vartheta)\delta^\varphi_\rho - (-\sin\vartheta\cos\vartheta)\delta^\varphi_\rho(\cot\vartheta)\delta^\vartheta_\sigma) - (\rho \leftrightarrow \sigma) \\
&= ([-\cos^2\vartheta + \sin^2\vartheta] + \cos^2\vartheta)\delta^\varphi_\rho\delta^\vartheta_\sigma - (\rho \leftrightarrow \sigma) = \sin^2\vartheta(\delta^\varphi_\rho\delta^\vartheta_\sigma - \delta^\varphi_\sigma\delta^\vartheta_\rho) \neq 0 \\
R^\varphi_{\vartheta\sigma\rho} &= (\partial_\sigma \Gamma^\varphi_{\rho\vartheta} - \Gamma^\varphi_{\rho\vartheta}\Gamma^\vartheta_{\sigma\vartheta} - \Gamma^\varphi_{\rho\varphi}\Gamma^\varphi_{\sigma\vartheta}) - (\rho \leftrightarrow \sigma) = (\partial_\sigma(\cot\vartheta)\delta^\varphi_\rho - (\cot\vartheta)\delta^\vartheta_\rho(\cot\vartheta)\delta^\varphi_\sigma) - (\rho \leftrightarrow \sigma) \\
&= ([-\sin^{-2}\vartheta]\delta^\varphi_\rho\delta^\vartheta_\sigma - \cot^2\vartheta\delta^\vartheta_\rho\delta^\varphi_\sigma) - (\rho \leftrightarrow \sigma) = -(\delta^\varphi_\rho\delta^\vartheta_\sigma - \delta^\varphi_\sigma\delta^\vartheta_\rho) \neq 0 \\
R^\varphi_{\varphi\sigma\rho} &= (\partial_\sigma \Gamma^\varphi_{\rho\varphi} - \Gamma^\varphi_{\rho\vartheta}\Gamma^\vartheta_{\sigma\varphi} - \Gamma^\varphi_{\rho\varphi}\Gamma^\varphi_{\sigma\varphi}) - (\rho \leftrightarrow \sigma) = (\delta^\vartheta_\rho\partial_\vartheta(\cot\vartheta)\delta^\vartheta_\rho - (\cot\vartheta)\delta^\varphi_\rho(-\sin\vartheta\cos\vartheta)\delta^\varphi_\rho) - (\rho \leftrightarrow \sigma) = 0
\end{aligned}$$

$\Rightarrow$ S^2 is not flat

Properties of the Riemann curvature tensor

- the first Bianchi identity

$$R^{\mu}{}_{\nu\rho\sigma} + R^{\mu}{}_{\rho\sigma\nu} + R^{\mu}{}_{\sigma\nu\rho} = 0 \qquad \Leftrightarrow \qquad R^{\mu}{}_{[\nu\rho\sigma]} = 0$$

- with the abbreviation $\nabla_\lambda X =: X_{;\lambda}$, the second Bianchi identity

$$R^{\mu}{}_{\nu\rho\sigma;\lambda} + R^{\mu}{}_{\nu\lambda\rho;\sigma} + R^{\mu}{}_{\nu\sigma\lambda;\rho} = 0 \qquad \Leftrightarrow \qquad R^{\mu}{}_{\nu[\rho\sigma;\lambda]} = 0$$

- with lowering the first index $R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^{\lambda}{}_{\nu\rho\sigma}$

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

- contraction gives the symmetric Ricci tensor and the Ricci scalar

$$R^{\lambda}{}_{\mu\lambda\nu} = g^{\lambda\rho} R_{\lambda\mu\rho\nu} = R_{\mu\nu} = R_{\nu\mu} \qquad g^{\mu\nu} R_{\mu\nu} = R$$

- this allows the decomposition $\qquad R_{\mu\nu\rho\sigma} = S_{\mu\nu\sigma\rho} + E_{\nu\mu\rho\sigma} + C_{\rho\sigma\mu\nu}$
  - into a scalar part $S_{\mu\nu\sigma\rho} = \frac{R}{n(n-1)}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})$
  - using the traceless Ricci tensor $S_{\mu\nu} = R_{\mu\nu} - \frac{R}{n} g_{\mu\nu}$ into a semi traceless part

$$E_{\mu\nu\sigma\rho} = \frac{1}{n-2}(g_{\mu\sigma}S_{\nu\rho} - g_{\mu\rho}S_{\nu\sigma} + g_{\nu\rho}S_{\mu\sigma} - g_{\nu\sigma}S_{\mu\rho})$$

  - and into the fully traceless Weyl tensor $C_{\mu\nu\sigma\rho}$

Curvature in general

- for a curve we can imagine, how it bends on the plane or in space
  - this is the extrinsic curvature
  - the intrinsic curvature of a curve is zero

  ⇒ it is similar to a straight line

- for a two dimensional surface we can imagine its bending in space
  - its intrinsic curvature is defined independently from the embedding
    - in our example of the spere S^2 we calculated the Riemann curvature tensor

$$R^{\vartheta}{}_{\varphi\vartheta\varphi} = -R^{\vartheta}{}_{\varphi\varphi\vartheta} = \sin^2\vartheta \qquad R^{\varphi}{}_{\vartheta\vartheta\varphi} = -R^{\varphi}{}_{\vartheta\varphi\vartheta} = -1$$

    - this gives the Ricci tensor $R_{\vartheta\vartheta} = R^{\varphi}{}_{\vartheta\varphi\vartheta} = 1$, $R_{\varphi\varphi} = R^{\vartheta}{}_{\varphi\vartheta\varphi} = \sin^2\vartheta$
    - and the Ricci scalar $R = g^{\vartheta\vartheta}R_{\vartheta\vartheta} + g^{\varphi\varphi}R_{\varphi\varphi} = r^{-2}\cdot 1 + r^{-2}\sin^{-2}\vartheta\cdot\sin^2\vartheta = \frac{2}{r^2}$

  ⇒ which gives the radius of the sphere ...

- for higher dimensional (hyper)surfaces the Ricci scalar is the generalisation of the curvature radius using the metric we can change our basis to make it orthonormal:

- defining a new set of local coordinates

$$e_{(m)}(x) = e^{\mu}_{m}(x)\partial_{(\mu)}$$

  - the greek indices μ indicate the natural basis $\partial_{(\mu)}$
    * obtained from the coordinate functions $X(x)$ on M
  - the latin indices m indicate the orthonormal basis $e_{(m)}(x)$
    * defined by the relation $g(e_{(m)}(x), e_{(n)}(x))(x) = \eta_{mn}$
    * the flat Minkovsky metric $\eta_{00} = 1$, $\eta_{ii} = -1$, and $\eta_{m\neq n} = 0$
- $e_{(m)}(x)$ is an orthonormal coordinate system
- $e^{\mu}_{m}(x)$ are called the "vierbein" or tetrad
  - its inverse $e^{m}_{\mu}(x)$ is defined by $e^{\mu}_{m}e^{m}_{\nu} = \delta^{\mu}_{\nu}$ and $e^{\mu}_{m}e^{n}_{\mu} = \delta^{n}_{m}$
  - we can write the metric tensor in natural coordinates as

$$g_{\mu\nu}(x) = e^{m}_{\mu}(x)e^{n}_{\nu}(x)\eta_{mn}$$

using the tetrad

- similar like for the natural coordinate basis $\partial_{(\mu)}$
  - we can introduce a dual basis in T_P^* by requiring $e^m(e_n) = \delta^m_n$
  - this dual basis can also be written as $e^m = e^m_\mu dx^{(\mu)}$
- with the tetrad we can change any index into the orthonormal frame
  - the vectors are just represented in a different basis
    - tensors can even have mixed components: $T^\mu_{\rho\sigma} = e^\mu_m T^m_{\rho\sigma} = e^\mu_m e^r_\rho T^m_{r\sigma} = e^\mu_m e^r_\rho e^s_\sigma T^m_{rs}$
- changing the coordinates $x \to x'$ changes components of a tensor by

$$T^{\mu'}_{\rho'\sigma'}(x') = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\rho}{\partial x^{\rho'}} \frac{\partial x^\sigma}{\partial x^{\sigma'}} T^\mu_{\rho\sigma}(x)$$

$\Rightarrow$ this is called a general coordinate transformation (GCT)

- changing the orthonormal basis $e_n \to e'_n$ to another orthonormal one
  - the metric does not change $\eta_{mn} = \eta'_{mn}$

  $\Rightarrow$ the transformation is a local Lorentz transformation (LLT) :

$$e_n(x) \to e'_n(x) = \Lambda^m{}_n(x) e_m(x)$$

a connection is not a tensor

⇒ a transformation of its indices might not lead to a connection

- parametrizing the connection in the orthonormal indices as

$$\nabla_\mu T^a_{\ b} = \partial_\mu T^a_{\ b} + \omega_\mu{}^a{}_c T^c_{\ b} - \omega_\mu{}^c{}_b T^a_{\ c}$$

- studying the covariant derivative of a vector helps:

$$\begin{aligned} \nabla_{dx} V &= dx^\mu (\nabla_\mu V)^c e_c = (\partial_\mu V^c + \omega_\mu{}^a{}_b V^b) dx^\mu \otimes (e_c^\sigma \partial_{(\sigma)}) = e_c^\sigma (\partial_\mu (e^c_\nu V^\nu) + \omega_\mu{}^c{}_b e^b_\rho V^\rho) dx^\mu \otimes \partial_{(\sigma)} \\ &= e_c^\sigma e^c_\nu ((\partial_\mu V^\nu) + e_a^\nu (\partial_\mu e^a_\rho) V^\rho + e_a^\nu e^b_\rho \omega_\mu{}^a{}_b V^\rho) dx^\mu \otimes \partial_{(\sigma)} \\ &= (\partial_\mu V^\nu + e_a^\nu (\partial_\mu e^a_\rho) V^\rho + e_a^\nu e^b_\rho \omega_\mu{}^a{}_b V^\rho) dx^\mu \otimes \partial_{(\nu)} = (\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho) dx^\mu \otimes \partial_{(\nu)} \end{aligned}$$

⇒ so $\Gamma^\nu_{\mu\rho} = e_a^\nu (\partial_\mu e^a_\rho) + e_a^\nu e^b_\rho \omega_\mu{}^a{}_b$ or $\omega_\mu{}^a{}_b = e^a_\lambda e^\rho_b \Gamma^\lambda_{\mu\rho} - e^\rho_b (\partial_\mu e^a_\rho)$

- multiplying with e^b_ν this can be written as the "tetrad posulate"

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega_\mu{}^a{}_b e^b_\nu = 0$$

- $\omega_\mu{}^a{}_b$ is called the spin connection
  - since it transports an index that transforms under Lorentz transformations
    - and Lorentz transformations act also on spinors

the spin connection

- transforms as a one-form under GCTs
- but inhomogenously under LLTs

$$\omega_\mu{}^a{}_b \to \omega'_\mu{}^a{}_b = \Lambda^a{}_c \Lambda^d{}_b \omega_\mu{}^c{}_d - \Lambda^c{}_b \partial_\mu \Lambda^a{}_c$$

- allows the implementation of a covariant exterior derivative:
  - by writing forms we can suppress the tensor index of the natural basis:
    - EM potential $A = A_\mu dx^\mu$, field strength $F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = \frac{1}{2} (dA)_{\mu\nu} dx^\mu dx^\nu$
  - having a vector valued (or group valued) one-form $A^a = A^a_\mu dx^\mu$,
  - we can extend the exterior derivative to a covariant exterior derivative

$$\begin{aligned} 2(\nabla A)^a &= (\nabla A)^a_{\mu\nu} dx^\mu dx^\nu = (dA)^a_{\mu\nu} dx^\mu dx^\nu + (\omega \wedge A)^a_{\mu\nu} dx^\mu dx^\nu \\ &= (\partial_\mu A^a_\nu + \omega_\mu{}^a{}_b A^b_\nu) dx^\mu dx^\nu \end{aligned}$$

  - writing the torsion as a vector valued two form:

$$\begin{aligned} T^a &= \tfrac{1}{2} T^a_{\mu\nu} dx^\mu dx^\nu = \tfrac{1}{2} e^a_\rho (\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}) dx^\mu dx^\nu = (\partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu) dx^\mu dx^\nu \\ &= (\nabla_\mu e^a_\nu) dx^\mu dx^\nu = \nabla e^a \end{aligned}$$

the spin connection

- calculating the curvature tensor as a tensor valued two form

$$\frac{1}{2}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V^a e_a^\rho dx^\mu dx^\nu \partial_{(\rho)} = e_a^\rho \nabla\nabla V^a \partial_{(\rho)} = e_a^\rho \nabla(dV^a + \omega_b^a V^b)\partial_{(\rho)}$$

$$= e_a^\rho (d(dV^a + \omega_b^a V^b) + \omega_c^a \wedge (dV^c + \omega_b^c V^b))\partial_{(\rho)}$$

$$= (d^2 V^a + (d\omega_b^a)V^b - \omega_b^a \wedge dV^b + \omega_c^a \wedge dV^c + \omega_c^a \wedge \omega_b^c V^b))e_a^\rho \partial_{(\rho)}$$

$$= (d\omega_b^a + \omega_c^a \wedge \omega_b^c)V^b e_a = R^a{}_b V^b e_a$$

$\Rightarrow$ gives the Maurer-Cartan structure equations

$$T^a = de^a + \omega_b^a \wedge e^b \qquad \text{and} \qquad R^a{}_b = d\omega_b^a + \omega_c^a \wedge \omega_b^c$$

- this allows an easy generalisation of the Bianchi identities ($R^\rho{}_{[\sigma\mu\nu]} = 0$):

$$\begin{aligned}\nabla T^a &= d(de^a + \omega_b^a \wedge e^b) + \omega_b^a \wedge (de^b + \omega_c^b \wedge e^c) \\ &= (d\omega_b^a) \wedge e^b - \omega_b^a \wedge (de^b) + \omega_b^a \wedge (de^b) + \omega_b^a \wedge \omega_c^b \wedge e^c = R^a{}_b \wedge e^b = \frac{1}{2}R^a{}_{b\mu\nu} \wedge e^b \wedge dx^\mu \wedge dx^\nu\end{aligned}$$

and ($R^\rho{}_{\sigma[\mu\nu;\lambda]} = 0$)

$$\begin{aligned}\nabla R^a{}_b &= d(d\omega_b^a + \omega_c^a \wedge \omega_b^c) + \omega_c^a \wedge (d\omega_b^c + \omega_d^c \wedge \omega_b^d) - (d\omega_d^a + \omega_c^a \wedge \omega_d^c) \wedge \omega_b^d \\ &= (d\omega_c^a) \wedge \omega_b^c - \omega_c^a \wedge (d\omega_b^c) + \omega_c^a \wedge (d\omega_b^c) + \omega_c^a \wedge (\omega_d^c \wedge \omega_b^d) - (d\omega_d^a) \wedge \omega_b^d - (\omega_c^a \wedge \omega_d^c) \wedge \omega_b^d = 0\end{aligned}$$

- metric compatibility $0 = \nabla_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega_{\mu}{}^c{}_a \eta_{cb} - \omega_{\mu}{}^c{}_b \eta_{ac} = -\omega_{\mu ba} - \omega_{\mu ab}$
  $\Rightarrow$ gives an antisymmetric spin connection $\omega_{\mu ab} = -\omega_{\mu ba}$