

1. General Relativity — vectors

What is a vector?

- a quantity that has a **size** and a **direction**
 - the **size** of a vector \vec{a} can be written as $|\vec{a}|$
 - $|\vec{a}|$ is usually a real number, but not necessarily positive
 - the **direction** is "optional"
- mathematical definition:
 - a **vector** is an **element** of a **vector space**
- examples
 - the real number x
 - the vector pointing from a point A to a point B
 - an n -tuple of numbers $\vec{x} = (x_1, x_2, \dots, x_n)$
 - the RGB color value (r, g, b)
- in **Geometric Algebra**
 - vectors are objects, that do not necessarily commute: $ab \neq ba$

1. General Relativity — vector space

mathematical definition:

- A vector space over a field (corpus) F is a set V with "+" and "*"
 - $(V, +)$ forms an abelian group: for $\vec{x}, \vec{y}, \vec{z} \in V$
 - * $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$
 - * $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
 - * there exists $\vec{0} \in V$ with $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in V$
 - * for all $\vec{x} \in V$ there exists $(-\vec{x}) \in V$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
 - $(F, +, *)$ forms a field (corpus)
 - * $(F, +)$ forms an abelian group
 - * $(F \setminus \{0\}, *)$ forms an abelian group
 - for $a, b \in F$
 - * $a * (\vec{x} + \vec{y}) = a * \vec{x} + a * \vec{y}$ and $(a + b) * \vec{x} = a * \vec{x} + b * \vec{x}$
 - * $a * (b * \vec{x}) = (a * b) * \vec{x}$
 - * $1 * \vec{x} = \vec{x}$ for "1" being the unit element of $(F \setminus \{0\}, *)$

1. General Relativity — vector space

examples:

- the real numbers over the real numbers
 - the real numbers are a field, so they form also an abelian group . . .
- vectors in \mathbf{R}^3 (three dimensional Euclidean space) over \mathbf{R}
- $m \times n$ real (complex) matrices over the real (complex) numbers
- tensors with arbitrary, but fixed index structure
 - example: all tensors of the form $t_i^j_k$
- real analytic functions in $[0, 1]$ over \mathbf{R}
- complex holomorphic functions over \mathbf{C}
- solutions to homogeneous linear differential equations over \mathbf{R} or \mathbf{C}

1. General Relativity — manifolds

loose mathematical definition:

- a manifold is a space that is **locally similar** to an Euclidean space
 - every point of an n-dimensional manifold has a neighborhood homeomorphic to an open subset of the n-dimensional space \mathbf{R}^n .
 - * "A is homeomorphic to B" = A can be continuously deformed to resemble B (actually, there has only to be a continuous invertible mapping between A and B)
 - ⇒ a ball is homeomorphic to a cube, but not to a torus
- ⇒ we can use our understanding of Euclidean space
 - including all our mathematical tools
 - and look, where the manifold differs from \mathbf{R}^n
- we can make local maps (charts) of the manifold
 - in regions where maps overlap, they should be compatible
 - compatible maps that cover the whole manifold form an atlas
- example: the surface of the earth
 - we need at least two maps to show the whole surface

1. General Relativity — manifolds

why manifolds?

- they generalize normal Euclidean space
 - example: the surface of the earth is not really Euclidean
- they preserve the local information
- ⇒ one can discuss causality
 - no need for "action at a distance"
- one can study study wave phenomena on a manifold
 - for that one needs differential equations – wave equations
 - ⇒ differentiable manifolds
- if we can measure distances and angles
 - ⇒ (Pseudo-) Riemannian manifolds
 - * they are also equipped with a metric

1. General Relativity — manifolds

differentiable manifolds

- have a globally defined differential structure
 - one can define differentiation similar to Euclidean space
 - * even multivariable differentiation
- allow a globally defined differentiable tangent space
 - one can define differentiable
 - * functions
 - * vectors
 - * tensors
- calculus on differential manifolds
 - ⇒ exterior derivative (Élie Cartan)
 - generalizes the differential of a function to forms of higher degree
- the calculus on differential manifolds is differential geometry

1. General Relativity — differential geometry

describes the properties of the manifold "from within"

- a smooth (i.e. differentiable) curve C on a manifold M can be understood as a smooth multidimensional function:

$$C : \mathbf{R} \rightarrow M$$

- example: the 3D-helix $C(t) = (r \cos t, r \sin t, bt)$
- studying the smooth curves through a point P :
 - differentiation with respect to t gives the tangent vector:

$$\vec{v}_P = \left. \frac{d}{dt} C(t) \right|_P = (-r \sin t, r \cos t, b)|_P$$

- all tangent vectors in P form the **tangent space** T_P
- the **tangent space** has the same dimension as M
 - but is isomorphic (identical in structure) to \mathbf{R}^n

1. General Relativity — tangent space

N dimensional tangent space T_P

- since we can reparametrize the curves $C(t) \rightarrow C(s) = C(t(s))$
 - the length of the tangent vector changes, but not its direction
- one can determine N linearly independent vectors
 \Rightarrow **basis vectors** of the tangent space T_P
- with coordinate functions $X_{(\mu)}(t)$ on the manifold
 - we can define a "natural" basis by

$$\vec{e}_{(\mu)} = \left. \frac{d}{dt} X_{(\mu)}(t) \right|_P = \text{"}\partial_{(\mu)}\text{"}$$

- any vector in T_P is a linear combination of the basis vectors:

$$\vec{A} = \sum_{\mu}^N A^{\mu} \vec{e}_{(\mu)} \quad \text{or} \quad A = A^{\mu} \partial_{(\mu)}$$

- A^{μ} are the (contravariant) components of the vector A

1. General Relativity — tangent space

N dimensional tangent space T_P

- using the coordinates $X_{(\mu)}$ on the manifold
 - we can define smooth functions of the manifold: $f(X) \in \mathbf{R}$
- ⇒ then **vectors** \vec{v} of the tangent space T_P can be seen as directional derivatives

$$D_{\vec{v}}f(\vec{x}) = \frac{d}{dt}f(\vec{x} + \vec{v}t) = \sum_{\mu} v^{\mu} \frac{\partial}{\partial x^{\mu}} f(\vec{x})$$

- again, the derivatives $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$ form "natural" basis directions
- one can define the cotangent space T_P^* as the dual to T_P :
 - every element $\omega \in T_P^*$ is a linear map $\omega : T_P \rightarrow \mathbf{R}$
 $\omega(aV + bW) = a\omega(V) + b\omega(W) \in \mathbf{R}$ for $V, W \in T_P$ and $\omega \in T_P^*$
 - and the dual vector space to T_P^* is T_P : $V : T_P^* \rightarrow \mathbf{R}$
 $V(a\omega + b\eta) = aV(\omega) + bV(\eta) \in \mathbf{R}$ for $\omega, \eta \in T_P^*$ and $V \in T_P$

1. General Relativity — tangent space

N dimensional cotangent space T_P^*

- using the basis vectors $\vec{e}_{(\mu)}$ of T_P

- we can define basis vectors $\vec{\theta}^{(\mu)}$ in T_P^* by

$$\vec{\theta}^{(\nu)}(\vec{e}_{(\mu)}) = \vec{e}_{(\mu)}(\vec{\theta}^{(\nu)}) = \delta_{\mu}^{\nu}$$

- and write a covector ω in terms of the N basis vectors $\vec{\theta}^{(\mu)}$:

$$\omega = \omega_{\mu} \vec{\theta}^{(\mu)}$$

- the map $T_P \times T_P^* \rightarrow \mathbf{R}$ can be expressed as:

$$\omega(V) = \omega_{\mu} V^{\nu} \vec{\theta}^{(\mu)}(\vec{e}_{(\nu)}) = \omega_{\mu} V^{\nu} \delta_{\nu}^{\mu} = \omega_{\mu} V^{\mu} \in \mathbf{R}$$

\Rightarrow one can use components **without specifying a basis**

* assuming that the bases of T_P and T_P^* are related by $\vec{\theta}^{(\nu)}(\vec{e}_{(\mu)}) = \delta_{\mu}^{\nu}$

- all T_P with $P \in M$ give the $2N$ dimensional **vector bundle** $T_P(M)$
- all T_P^* with $P \in M$ give the $2N$ dimensional **cotangent bundle** $T_P^*(M)$

1. General Relativity — tangent space

example $T_P(S^2)$: the 2-sphere S^2 with \mathbf{R}^2 attached to each point

- looking at the embedding in \mathbf{R}^3 we can choose coordinates (x, y, z)
 - for the sphere they have to satisfy $x^2 + y^2 + z^2 = r^2 = 1$
 - going to spherical coordinates we have on the sphere $(r = 1, \vartheta, \varphi)$
 - * with $x = \sin \vartheta \cos \varphi$, $y = \sin \vartheta \sin \varphi$, and $z = \cos \vartheta$
- ⇒ so for a coordinate patch (a map) we can use (ϑ, φ)

- the point $P_0 = (\vartheta_0, \varphi_0)$ in S^2 is embedded in \mathbf{R}^3 :
 - considering the curves through P_0

$$C_1(s) = (\sin \vartheta_0 \cos s, \sin \vartheta_0 \sin s, \cos \vartheta_0)$$

$$C_2(t) = (\sin t \cos \varphi_0, \sin t \sin \varphi_0, \cos t)$$

- the tangent plane at P_0 has tangent vectors in \mathbf{R}^3

$$\vec{e}_{(1)} = \frac{d}{ds} C_1(s) \Big|_{P_0} = \sin \vartheta_0 (-\sin \varphi_0, \cos \varphi_0, 0)$$

$$\vec{e}_{(2)} = \frac{d}{dt} C_2(t) \Big|_{P_0} = (\cos \vartheta_0 \cos \varphi_0, \cos \vartheta_0 \sin \varphi_0, -\sin \vartheta_0)$$

- since $\vec{e}_{(1)}$ and $\vec{e}_{(2)}$ are linearly independent: $\nexists \lambda$ with $\vec{e}_{(1)} = \lambda \vec{e}_{(2)}$
- ⇒ they form a basis in T_{P_0}

- using this basis, we can write any vector V in T_{P_0} as $V = V^1 \vec{e}_{(1)} + V^2 \vec{e}_{(2)} = V^i \vec{e}_{(i)}$

⇒ the coordinates of V in $T_P(S^2)$ are $(\vartheta_0, \varphi_0, V^1, V^2)$

1. General Relativity — tangent space

example $T_P(S^2)$: the 2-sphere S^2 with \mathbf{R}^2 attached to each point

- picking different curves through P_0 is the same as changing the basis of T_{P_0}
 - $V \rightarrow V'$ or $(\vartheta_0, \varphi_0, V^1, V^2) \rightarrow (\vartheta_0, \varphi_0, V'^1, V'^2)$
 - this change is a normal coordinate transformation in \mathbf{R}^2
 - * which can also stretch and rotate T_{P_0}
- going to a point $P_1 = (\vartheta_1, \varphi_1)$
 - is a normal translation on S^2
 - but it can also change the basis of T_P : $(\vec{e}_{(1)}, \vec{e}_{(2)}) \rightarrow (\vec{e}'_{(1)}, \vec{e}'_{(2)})$
 - in \mathbf{R}^2 (and \mathbf{R}^n) we have an understanding, what is parallel
 - but $\vec{e}_{(i)}$ and $\vec{e}'_{(i)}$ are not necessarily parallel
 - * in the embedding space \mathbf{R}^3 they would be

$$\begin{array}{lll} \vec{e}_{(1)} = s_{\vartheta_0}(-s_{\varphi_0}, c_{\varphi_0}, 0) & \text{not parallel} & \vec{e}'_{(1)} = s_{\vartheta_1}(-s_{\varphi_1}, c_{\varphi_1}, 0) \\ \vec{e}_{(2)} = (c_{\vartheta_0}c_{\varphi_0}, c_{\vartheta_0}s_{\varphi_0}, -s_{\vartheta_0}) & \text{not parallel} & \vec{e}'_{(2)} = (c_{\vartheta_1}c_{\varphi_1}, c_{\vartheta_1}s_{\varphi_1}, -s_{\vartheta_1}) \end{array}$$

- * in $T_P(S^2)$ we only see

$$\begin{array}{lll} \vec{e}_{(1)} = (\vartheta_0, \varphi_0, 1, 0) & \dots & \vec{e}_{(2)} = (\vartheta_0, \varphi_0, 0, 1) \\ \vec{e}'_{(1)} = (\vartheta_1, \varphi_1, 1, 0) & \dots & \vec{e}'_{(2)} = (\vartheta_1, \varphi_1, 0, 1) \end{array}$$

- how can we compare them from inside?
 - we need a **connection**

1. General Relativity — connections

connections allow the definition of parallel transport on $T_P(M)$

- for $P_0, P_1 \in M$ the affine connection relates T_{P_0} and T_{P_1}
 - so that tangent vector fields can be differentiated
 - * i.e. compared between the points P_0 and P_1
 - * this comparison usually uses the concepts of pullback and pushforward
 - * these concepts become too mathematically abstract for this lecture ...
- for our example it means, that by using an affine connection
 - we can effectively reduce the dimension of $T_P(M)$ back to T_P (or M)
 - and the basis vectors in T_P are smooth functions of the point:

$$\vec{e}_{(1)}(\vartheta, \varphi) \quad \text{and} \quad \vec{e}_{(2)}(\vartheta, \varphi)$$

- the Cartan connection uses the Lie group structure of M
 - and transports (coordinate) frames without specifying a metric
 - * uses the exterior derivative
- the Levi-Civita connection uses the additional structure of a metric
 - ⇒ Riemannian manifolds ... will be assumed for the rest of this lecture

1. General Relativity — exterior derivative

formalizes the known differential of a multiparametric function

$$df(x, y, z) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

- choosing coordinates $(x, y, z) = (x^1, x^2, x^3)$
- the quantities dx , dy , and dz are understood as covectors or **1-forms**
 - forming a basis of T_P^* with $\vec{\theta}^{(\nu)} = dx^\nu$
- the wedge product " \wedge " defines the multiplication of forms
 - it is antisymmetric: $dx \wedge dy = -dy \wedge dx$
 - \Rightarrow it is nilpotent: $dx \wedge dx = 0$
 - for obvious differentials dx the \wedge can be omitted: $dx dy := dx \wedge dy$
 - a one-form in this natural basis: $\omega = \omega_\mu dx^\mu$
 - a p -form in this natural basis: $\Omega = \Omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$
 - * obviously, $\Omega_{\mu_1 \dots \mu_p}$ is completely antisymmetric in its indices
 - * and $p \leq N$, the dimension of M

1. General Relativity — exterior derivative

features of the exterior derivative d

- increases the grade of a form
 - defining the smooth functions f or x^μ as 0-forms
 - we have 1-forms df or dx^μ
 - a 2-form $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$ gives a 3-form $dF = (\partial_\rho F_{\mu\nu})dx^\rho \wedge dx^\mu \wedge dx^\nu$
 - for a N -form $I = I_{\mu_1 \dots \mu_N} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_N}$ we get $dI = 0$

- defines the natural basis by

$$df(V) = (\partial_\mu f) dx^\mu (V^\nu \partial_\nu) = V^\nu (\partial_\mu f) \delta_\nu^\mu = V^\mu \partial_\mu (f) = V(f)$$

- follows the Leibnitz rule
 - for a p -form α and a q -form β

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

- is useful for the discussion of fiber bundles $G(M)$
 - with the Lie group G defined at every point of M
 - the tangent bundle $T_P(M)$ is a fiber bundle with the tangent space T_P as the structure given at each point in M

- all forms on M form a vector space of dimension 2^N

1. General Relativity — Lie derivative

vector fields as derivatives of functions

- $df(V) = V(f)$ is again a function on M for $V \in T_P$
- for $X, Y \in T_P$ we can define the **Lie derivative** or **Lie bracket**

$$\mathcal{L}_X(Y)(f) = [X, Y](f) := X(Y(f)) - Y(X(f)) \in \mathbf{R}$$

- the **Lie bracket**
 - is bilinear and antisymmetric
 - fulfills the **Jacobi identity**
$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$
 - $\mathcal{L}_X(Y)$ can be seen as the directional of Y along X
 - \Rightarrow it allows to define a parallel transport
 - * for $[X, Y](f) = 0$, $Y(f)$ stays constant along the flow of X
- the **Lie bracket** endows M with the algebraic structure of a Lie algebra

1. General Relativity — the metric

mathematical definition:

- a metric in M is a symmetric, bilinear, non-degenerate function

$$g_P : T_P \times T_P \rightarrow \mathbf{R}$$

- the metric acts on (contravariant) vectors (tensor indices)
- symmetric means $g(X, Y) = g(Y, X) \in \mathbf{R}$ for $X, Y \in T_P$
- bilinear means for $X, Y, Z \in T_P$ and $a, b, c \in \mathbf{R}$

$$g(aX + bY, cZ) = ac g(X, Z) + bc g(Y, Z) \in \mathbf{R}$$

- non-degenerate: $g(X, Y) = 0$ for all $Y \in M$ only if $X = 0$
- acting on the basis vectors $\vec{e}_{(\mu)}$ gives the **metric tensor**

$$g_{\mu\nu}(P) = g(\vec{e}_{(\mu)}, \vec{e}_{(\nu)}) \Big|_P$$

- the metric allows length and angle measurements

$$\|X\| := \sqrt{|g(X, X)|} \quad \text{and} \quad \cos \varphi = \frac{g(X, Y)}{\|X\| * \|Y\|}$$

1. General Relativity — the metric

examples for the metric

- in Euclidean space the metric is the normal dot-product:

$$\vec{a} \cdot \vec{b} = g_{jk} a^j b^k = \delta_{jk} a^j b^k \quad \text{so} \quad g_{j=k} = 1 \quad \text{and} \quad g_{j \neq k} = 0$$

- we can use this metric of the embedding to get the **induced** metric in S^2

$$g_{jk} = g(\vec{e}_{(j)}, \vec{e}_{(k)}) = \vec{e}_{(j)} \cdot \vec{e}_{(k)} \quad \text{so} \quad g_{11} = s_{\vartheta_0}^2 \quad g_{12} = g_{21} = 0 \quad g_{22} = 1$$

⇒ as we can see, the metric tensor depends on the position

- in Minkovsky space we had $g_{00} = 1$, $g_{ii} = -1$, and $g_{\mu \neq \nu} = 0$

- we can generalize the line element

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = g_{\mu\nu} \Delta x^\mu \otimes \Delta x^\nu$$

- the "differentials" Δx here are **not** multiplied with their "natural" wedge product

- one can define the **inverse metric** $g^{\mu\nu}$ by

$$g^{\mu\nu} g_{\mu\rho} = \delta_\rho^\nu$$

- the inverse metric is the metric of the cotangent space