Review of Chapter 5 from *Introduction to Plasma Physics: with Space, Laboratory and Astrophysical Applications* by Gurnett and Bhattacharjee

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Kinetic theory and the moment equations

- There are a lot of particles in a plasma.
- Description of the motion of each individual particle is not practical.
- We will use a statistical description – kinetic theory.

The distribution function

- The distribution function is defined by

$$\mathrm{d}N = f\left(\vec{r}, \vec{v}, t\right)\, \mathrm{d}^3 r \mathrm{d}^3 v$$

  being the number of particles in a phase space volume $\mathrm{d}^3 r \mathrm{d}^3 v$.
- We have to assume that the number of particles is large, so that even in a quite small phase space volume there are enough particles for the statistical description to hold.
- The distribution function is normalized so that

$$N = \int \mathrm{d}^3 r \int \mathrm{d}^3 v f\left(\vec{r}, \vec{v}, t\right)$$

  is the total number of particles.
- We will use $f_s\left(\vec{r}, \vec{v}, t\right)$ for each particle species s in the plasma.

Averages

- We can calculate various averages:

$$n_s = \int \mathrm{d}^3 v \, f_s \left(\vec{r}, \vec{v}, t\right);$$

$$U_{s,i} = \frac{1}{n_s} \int \mathrm{d}^3 v \, v_i f_s \left(\vec{r}, \vec{v}, t\right);$$

$$W_s = \int \mathrm{d}^3 v \, \frac{1}{2} m_s v^2 f_s \left(\vec{r}, \vec{v}, t\right);$$

$$P_{s,ij} = \int \mathrm{d}^3 v \, m_s \left(v - U_{s,i}\right) \left(v - U_{s,j}\right) f_s \left(\vec{r}, \vec{v}, t\right);$$

Examples

- A Maxvellian velocity distribution

$$f = n_0 \left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{3/2} \exp\left[-\frac{m\left(\vec{v} - \vec{U}_0\right)^2}{2k_{\mathrm{B}} T}\right]$$

  results in a pressure tensor

$$P_{ij} = n_0 k_{\mathrm{B}} T \delta_{ij}.$$

- A bi-Maxvellian distribution

$$f = n_0 \left(\frac{m}{2\pi k_{\mathrm{B}} T_\perp}\right) \left(\frac{m}{2\pi k_{\mathrm{B}} T_\parallel}\right)^{1/2} \exp\left[-\frac{m v_\perp^2}{2k_{\mathrm{B}} T_\perp}\right] \exp\left[-\frac{m v_\parallel^2}{2k_{\mathrm{B}} T_\parallel}\right]$$

  results in a pressure tensor

$$\overleftrightarrow{P} = \begin{bmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{bmatrix}.$$

The continuity equation

- Let us assume that $\rho\left(\vec{z}, t\right)$ describes a distribution function with $\vec{z}$ being an abstract n-dimensional vector. Then

$$\int_V \mathrm{d}^n z \, \rho\left(\vec{z}, t\right) = N$$

is the total number of particles in a phase space volume V.

- If states in the phase space neither appear nor disappear out of nowhere, then

$$\frac{\partial}{\partial t} \int_{\Delta V} \mathrm{d}^n z \, \rho\left(\vec{z}, t\right) = -\oint \mathrm{d}\vec{S} \cdot \dot{\vec{z}} \rho = -\int_{\Delta V} \mathrm{d}^n z \, \nabla \cdot \left(\dot{\vec{z}} \rho\right).$$

Since ΔV is arbitrary, then

$$\frac{\partial}{\partial t} \rho\left(\vec{z}, t\right) + \nabla \cdot \left(\dot{\vec{z}} \rho\right) = 0.$$

The Boltzmann and Vlasov equations

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The Boltzmann and Vlasov equations

- Taking $\rho \to f, \vec{z} \to \vec{r}, \vec{v}, \dot{\vec{r}} = \vec{v}, \dot{\vec{v}} = \vec{F}/m$ and assuming that $\vec{F}$does not depend on $\vec{v}$ we obtain

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{\vec{F}}{m} \cdot \nabla_{\vec{v}} f = 0.$$

- The Vlasov equation results if we ignore collisions and use the Lorentz force:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{q}{m} \left(\vec{E} + \left[\vec{v} \times \vec{B} \right] \right) \cdot \nabla_{\vec{v}} f = 0.$$

- The Boltzmann equations results if we additionally account for short range interactions as collisions:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{\vec{F}}{m} \cdot \nabla_{\vec{v}} f = \frac{\delta_c f}{\delta t}.$$

Solutions based on constants of the motion

- The Vlasov equation can be written as $\frac{\mathrm{d}f_s}{\mathrm{d}t}$[1].
- Then any function of the constants of the motion, $f_s\left(C_1\left(\vec{r},\vec{v}\right),C_2\left(\vec{r},\vec{v}\right),\ldots\right)$, satisfies the Vlasov equation.
- For example, for particles moving in a stationary electrostatic potential $\Phi\left(\vec{r}\right)$ we obtain that the Maxwell-Boltzmann distribution satisfies the Vlasov equation:

$$f\left(\vec{r},\vec{v}\right)=n_0\left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{3/2}\exp\left[-\frac{\frac{1}{2}mv^2+q\Phi\left(\vec{r}\right)}{k_\mathrm{B}T}\right].$$

[1] Formally we would need the material derivative, as here we interchange $\vec{v}$ from a velocity field to a velocity of a particle.

The moment equations

- The idea is to multiply the Boltzmann equation by powers of the velocity and integrating over the velocity space.
- Involves some writing and using of the Gauss' theorem to discad the "surface" terms.
- For the zeroth moment,

$$\frac{\partial n_s}{\partial t} + \nabla \cdot \left(n_s \vec{U}_s \right) = 0.$$

For the first moment,

$$m_s n_s \left(\frac{\partial}{\partial t} \vec{U}_s + \left(\vec{U}_s \cdot \nabla \right) \vec{U}_s \right) = n_s e_s \left(\vec{E} + \left[\vec{U}_s \times \vec{B} \right] \right) - \nabla \cdot \overleftrightarrow{P} + \frac{\delta_c \vec{p}_s}{\delta t}.$$

Here $\frac{\delta_c \vec{p}_s}{\delta t}$ is the average rate of change of the momentum per unit volume due to collisions.

The moment equations

- For the second moment, the issue is complicated since we would need equations for the entire pressure tensor.
- If we restrict ourselves to the trace, we obtain

$$\frac{\partial W_s}{\partial t} + \nabla \cdot \vec{Q}_s - \vec{E} \cdot \vec{J}_s = \int \mathrm{d}^3 v \, \frac{1}{2} m_s v^2 \frac{\delta_\mathrm{c} f_s}{\delta t}.$$

  Here

$$\vec{Q}_s = \int \mathrm{d}^3 v \, \frac{1}{2} m v^2 \vec{v}$$

  is the kinetic energy flux.

The closure problem

- Consider again the Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{\vec{F}}{m} \cdot \nabla_{\vec{v}} f = \frac{\delta_c f}{\delta t}.$$

- The term $\vec{v} \cdot \nabla f$ has $\vec{v}$, thus the equation for the nth moment will involve the $(n+1)$st moment.
- There will always be more unknowns than equations.
- To close the system of equations, we need to specify the $(n+1)$st moment in terms of the lower moments.
- Usually, this is done by relating the pressure tensor $\overleftrightarrow{P}_s$ to the number density n_s.

The cold plasma equation of state

- The temperature is assumed to be so low that the pressure is negligible, thus $\overleftrightarrow{P}_s = 0$ for any n_s.
- Then

$$m_s n_s \left(\frac{\partial}{\partial t} \vec{U}_s + \left(\vec{U}_s \cdot \nabla \right) \vec{U}_s \right) = n_s e_s \left(\vec{E} + \left[\vec{U}_s \times \vec{B} \right] \right) + \frac{\delta_{\mathrm{c}} \vec{p}_s}{\delta t}.$$

- If the collision term is zero, then

$$m_s n_s \left(\frac{\partial}{\partial t} \vec{U}_s + \left(\vec{U}_s \cdot \nabla \right) \vec{U}_s \right) = n_s e_s \left(\vec{E} + \left[\vec{U}_s \times \vec{B} \right] \right).$$

which is tagged and followed as a function of t. In some cases, this distinction can be important because the convective derivative, $\mathrm{d}\mathbf{U}_s/\mathrm{d}t$, consists of two terms, i.e.,

$$\frac{\mathrm{d}\mathbf{U}_s}{\mathrm{d}t} = \frac{\partial \mathbf{U}_s}{\partial t} + (\mathbf{U}_s \cdot \nabla)\mathbf{U}_s. \tag{5.4.38}$$

In the case of small-amplitude waves in a fluid at rest (i.e., $\mathbf{U}_{s0} = 0$), the second term on the right can be ignored, since it involves the product of two first-order quantities, $(\mathbf{U}_{s1} \cdot \nabla)\mathbf{U}_{s1}$, which is small compared with the first-order quantity $\partial \mathbf{U}_{s1}/\partial t$. We then have $\mathrm{d}\mathbf{U}_{s1}/\mathrm{d}t \simeq \partial \mathbf{U}_{s1}/\partial t$. The operator substitution $\mathrm{d}/\mathrm{d}t \rightarrow (-\mathrm{i}\omega)$ is then a good first-order approximation. On the other hand, if the fluid is moving, i.e., $\mathbf{U}_{s0} \neq 0$, then the correct linearized form for the operator $\mathrm{d}\mathbf{U}_{s1}/\mathrm{d}t$ is

$$\frac{\mathrm{d}\mathbf{U}_{s1}}{\mathrm{d}t} = \frac{\partial \mathbf{U}_{s1}}{\partial t} + (\mathbf{U}_{s0} \cdot \nabla)\mathbf{U}_{s1}. \tag{5.4.39}$$

After Fourier transforming, this equation becomes

$$(-\mathrm{i}\omega')\tilde{\mathbf{U}}_{s1} = (-\mathrm{i}\omega + \mathrm{i}\mathbf{k} \cdot \mathbf{U}_{s0})\tilde{\mathbf{U}}_{s1}, \tag{5.4.40}$$

where ω' is the frequency in a frame of reference moving with the particle. From

The adiabatic equation of state

- Assuming no heat flows, from the ideal gas law, $PV = Nk_{\mathrm{B}}T$ and the first law of thermodynamics, $\mathrm{d}Q = \mathrm{d}U + P\mathrm{d}V$, we can obtain a relationship $PV^{\gamma} = \text{const}$.
- Here $\gamma = C_{\mathrm{P}}/C_{\mathrm{V}}$.
- The adiabatic equation of state applies to situations where the gas is compressed so rapidly that there is not enough time for heat to flow.
- For this equation of state, it is assumed that the pressure tensor is isotropic, $\overleftrightarrow{P}_s = \overleftrightarrow{1} P_s$.
- From statistical mechanics it is known that $\gamma = (f + 2)/2$, with f being the number of degrees of freedom.

The Chew–Goldberger–Low (CGL) equation of state

- If collision rate an a magnetized plasma is too small to transfer momentum effectively between the parallel and perpendicular directions, an isotropic velocity distribution cannot be maintained.
- CGL equation of state assumes that

$$\overleftrightarrow{P}_s = \begin{bmatrix} P_{s\perp} & 0 & 0 \\ 0 & P_{s\perp} & 0 \\ 0 & 0 & P_{s\parallel} \end{bmatrix}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P_{s\perp}}{n_s B}\right) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P_{s\parallel} B^2}{n_s^3}\right) = 0.$$

- The CGL equation of state can be motivated by the first and second adiabatic invariants.

Electron and ion pressure waves

- We assume the adiabatic equation of state,

$$\overleftrightarrow{P}_s = P_{s0}\left(\frac{n_s}{n_{s0}}\right)^{\gamma}\overleftrightarrow{1},$$

and

$$P_s = P_{s0} + P_{s1}, \quad n_s = n_{s0} + n_{s1}, \quad \vec{U}_s = \vec{U}_{s1}, \quad \vec{E} = \vec{E}_1.$$

- We obtain

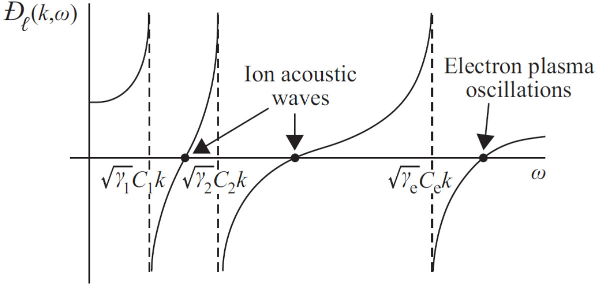
$$\begin{bmatrix} -c^2k^2 + \omega^2 - \omega_{\mathrm{p}}^2 & 0 & 0 \\ 0 & -c^2k^2 + \omega^2 - \omega_{\mathrm{p}}^2 & 0 \\ 0 & 0 & \omega^2 - \sum_s \frac{\omega_{\mathrm{p}s}^2}{1-\gamma_s C_s^2\left(k^2/\omega^2\right)} \end{bmatrix} \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0.$$

Here $C_s^2 = P_{s0}/(m_s n_{s0})$.

The longitudinal mode

- The dispersion relation can be written as

$$\mathcal{D}_\ell(k,\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \gamma_s C_s^2 k^2}$$

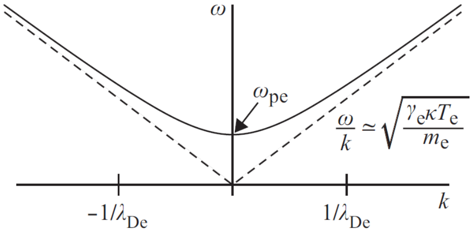


- The root associated with the electrons is called the Langmuir mode.

The Langmuir mode

- Since the electron plasma frequency is much greater than the characteristic frequencies of the ions, we can ignore the ion terms. Then, after rearranging the terms,

$$\omega^2 = \omega_{pe}^2 \left(1 + \gamma_e \lambda_{De}^2 k^2\right).$$

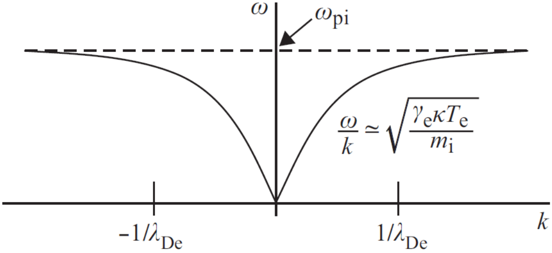


- The phase velocity is formally identical to that of an acoustic wave in an ordinary gas, but, apparently, Langmuir waves are strongly damped for large wavenumbers.

The Ion Acoustic mode

- We assume that the ion temperature is small, thus $C_i = 0$. We also assume that the phase velocity is much smaller than the electron thermal velocity, thus $\omega^2 \ll \gamma_e C_e^2 k^2$. Then

$$\omega^2 = \frac{1}{1 + \gamma_e \lambda_{De}^2 k^2} \left(\frac{\gamma_e k_B T_e}{m_i} \right) k^2.$$



Comparison of motion

- For the Lanngmuir mode

$$\frac{\tilde{U}_{\rm e}}{\tilde{U}_{\rm i}} \simeq \frac{m_{\rm i}}{m_{\rm e}}.$$

- For the ion acoustic mode

$$\frac{\tilde{U}_{\rm e}}{\tilde{U}_{\rm i}} \simeq 1 + \frac{m_{\rm e}}{m_{\rm i}}.$$

The Lorentz gas model

- It is assumed that the particles are scattered by fixed immobile scattering centers. Then we can write

$$\frac{\delta_c \vec{p}_s}{\delta t} = -\nu_s m_s n_s \vec{U}_s.$$

- This model works well for electron scattering by neutral particles.
- For ion-neutral collisions the isotropic approximation is that that good, since ions transfer some momentum to the neutral particles.
- Using the Lorentz model we can obtain the velocity

$$m_s n_s \frac{\partial \vec{U}_s}{\partial t} = e_s n_s \vec{E}_0 - \nu_s m_s n_s \vec{U}_s \quad \Rightarrow \quad \vec{U}_s = \frac{e_s \vec{E}_0}{m_s \nu_s} \left(1 - \mathrm{e}^{-\nu_s t}\right)$$

and then the conductivity:

$$\vec{J}_s = n_s e_s \vec{U}_{s0} = \underbrace{\left(\frac{n_s e_s^2}{m_s \nu_s}\right)}_{\sigma_s} \vec{E}_0.$$

Pedersen and Hall Conductivities

- It there is a magnetic field, the situation complicates a bit, but the conductivity tensor can be obtained,

$$\overleftrightarrow{\sigma} = \begin{bmatrix} \sigma_\perp & \sigma_\mathrm{H} & 0 \\ \sigma_\mathrm{H} & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{bmatrix}$$

  with

$$\sigma_\perp = \sum_s \frac{\sigma_s}{\left(1 + \omega_{\mathrm{c}s}^2/\nu_s^2\right)}, \quad \sigma_\mathrm{H} = \sum_s \frac{\sigma_s\left(\omega_{\mathrm{c}s}/\nu_s\right)}{\left(1 + \omega_{\mathrm{c}s}^2/\nu_s^2\right)}, \quad \sigma_\parallel = \sum_s \sigma_s .$$

  Here $\sigma_\parallel$ and $\sigma_\perp$ are the parallel and perpendicular Pederson conductivities and σ_H is the Hall conductivity.

Ambipolar diffusion

- We consider the diffusion of electrons and one species of positive ions using the Lorentz gas model.
- We can combine the continuity equations for electrons and ions:

$$\nabla \cdot \left(n \left(\vec{U}_{\mathrm{e}} - \vec{U}_{\mathrm{i}} \right) \right) = -\frac{\partial}{\partial t} \left(n_{\mathrm{e}} - e_{\mathrm{i}} \right)$$

  with $n = n_{\mathrm{e}} \simeq n_{\mathrm{i}}$. If the plasma is quasi-neutral, then the LHS of this equation should be 0.
- Often the only solution is given by $\vec{U}_{\mathrm{e}} = \vec{U}_{\mathrm{i}}$. If this is the case, we have ambipolar diffusion.
- The ambipolar diffusion is described by

$$\frac{\partial n}{\partial t} = \nabla^2 \left(\bar{\kappa}_{\mathrm{a}} n \right), \qquad \bar{\kappa}_{\mathrm{a}} = \frac{\bar{\kappa}_{\mathrm{e}} \bar{\mu}_{\mathrm{i}} + \bar{\kappa}_{\mathrm{i}} \left| \bar{\mu}_{\mathrm{e}} \right|}{\bar{\mu}_{\mathrm{i}} + \left| \bar{\mu}_{\mathrm{e}} \right|}.$$