Review of Chapter 4 from *Introduction to Plasma Physics: with Space, Laboratory and Astrophysical Applications* by Gurnett and Bhattacharjee

Andrius Gelžinis

Department of Molecular Compound Physics, Center for Physical Sciences and Technology

Institute of Chemical Physics, Faculty of Physics, Vilnius University

2024 10 04

Waves in cold plasmas

- We will consider small-amplitude waves (so everything can be linearized).
- A cold plasma means that particles are initially at rest, no thermal motions. Thus, Doppler shifts caused by thermal motion can be ignored.
    - I guess that cold plasma is a rather limited model, since we should assume energetic particles in plasmas.
- A cold plasma has no pressure, thus no sound waves. Also there are no instabilities.

Fourier representation of functions

- In this book the Fourier transform convention is

$$f\left(x\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}k\,\tilde{f}\left(k\right)\mathrm{e}^{\mathrm{i}kx}.$$

- A function of spatial coordinates and time can thus be represented as

$$f\left(\vec{r},t\right)=\frac{1}{\left(2\pi\right)^{2}}\int\mathrm{d}^{3}k\int_{-\infty}^{\infty}\mathrm{d}t\,\tilde{f}\left(\vec{k},\omega\right)\mathrm{e}^{\mathrm{i}\left(\vec{k}\cdot\vec{r}-\omega t\right)}.$$

- $\vec{k}$ is the wave vector. Plane of constant phase is described by $\vec{k}\cdot\vec{r}-\omega t=\text{const}$. The velocity with which the planes of constant phase move is called the phase velocity is $\vec{v}_{\mathrm{p}}=\frac{\omega}{k}\hat{k}$.

Dispersion relation

- Using the Fourier transform a linear differential equation can be transformed to a linear algebraic equation by

$$\frac{\partial}{\partial t} \rightarrow -\mathrm{i}\omega,$$

$$\nabla \rightarrow \mathrm{i}\vec{k}.$$

- Thus,

$$\mathcal{D}\left(\nabla, \partial/\partial t\right) f = 0 \quad \Rightarrow \quad \mathcal{D}\left(\mathrm{i}\vec{k}, -\mathrm{i}\omega\right) \tilde{f} = 0.$$

- Non-trivial solution for $\tilde{f}$ exists only if $\mathcal{D}\left(\mathrm{i}\vec{k}, -\mathrm{i}\omega\right) = 0$. This condition, then written as $\mathcal{D}\left(\vec{k}, \omega\right) = 0$ is called the dispersion relation. It gives a relationship between ω and $\vec{k}$. In many cases there are multiple discrete roots: $\omega = \omega_\alpha$ with $\alpha = 1, 2, \ldots, N$. The roots represent the normal modes.

Dispersion relation

- For a simple wave equation, we have

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0 \quad \Rightarrow \quad \left[-k^2 + \frac{\omega^2}{c^2}\right]\tilde{f} = 0.$$

  Thus $\mathcal{D}(k, \omega) = 0$ has two roots, $\omega = \pm ck$.

- In general,

$$f(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \sum_{\alpha=1}^{N} \int \mathrm{d}^3 k \tilde{f}_\alpha\left(\vec{k}\right) \exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r} - \omega_\alpha\left(\vec{k}\right) t\right]\right).$$

  The roots $\omega_\alpha\left(\vec{k}\right)$ determine the propagation speed of the individual plane waves. Since they tend to propagate at different speeds, the wave packet tends to spread out in space with increasing time. This spreading is called *dispersion*.

Dispersion relation

- For a vector wave field $\vec{f}$, we have a linear system of homogeneous differential equations

$$\hat{\mathcal{D}}\left(\nabla, \partial/\partial t\right) \cdot \vec{f} = 0 \quad \Rightarrow \quad \hat{\mathcal{D}}\left(\mathrm{i}\vec{k}, -\mathrm{i}\omega\right) \cdot \tilde{\vec{f}} = 0.$$

- A non-trivial solutions exists if and only if the determinant of the matrix is zero:

$$\mathcal{D}\left(\vec{k}, \omega\right) = \mathrm{Det}\left[\hat{\mathcal{D}}\left(\mathrm{i}\vec{k}, -\mathrm{i}\omega t\right)\right] = 0.$$

- Each root of the dispersion relation has a corresponding eigenvector $\tilde{\vec{f}}$, which characterizes the field geometry of the propagating wave.

Group velocity

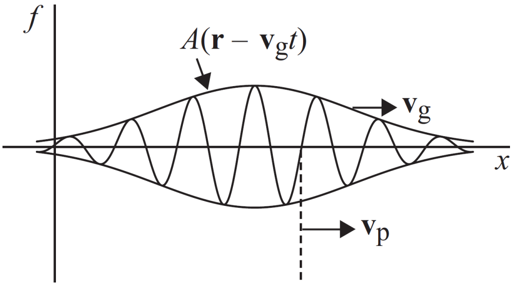
- If the Fourier image of the wave is sharply peaked around a particular wave number, the shape of the wave envelope of the wave packet is preserved to a first approximation. The velocity at which the envelope moves is called the *group velocity*.
- Consider

$$\begin{aligned}
&\exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r}-\omega\left(\vec{k}\right)t\right]\right)\\
&\quad\approx\exp\left(\mathrm{i}\left[\vec{k}\cdot\vec{r}-\left(\omega_0+\nabla_{\vec{k}}\omega\cdot\left(\vec{k}-\vec{k}_0\right)\right)t\right]\right)\\
&=\exp\left(\mathrm{i}\left[\vec{k}_0\cdot\vec{r}-\omega_0 t\right]\right)\exp\left(\mathrm{i}\left[\left(\vec{k}-\vec{k}_0\right)\cdot\left(\vec{r}-\nabla_{\vec{k}}\omega\cdot t\right)\right]\right).
\end{aligned}$$

- Thus, $\vec{v}_{\mathrm{g}}=\nabla_{\vec{k}}\omega$.

Group velocity

- We can write
$$A\left(\vec{r}-\vec{v}_{\mathrm{g}}t\right)\mathrm{e}^{\mathrm{i}\left(\vec{k}_0\cdot\vec{r}-\omega_0 t\right)}.$$



- Sometimes it will be useful to deal with a dimensionless index of refraction, $\vec{n}=c\vec{k}/\omega$. We will often express it as $\vec{n}\,(\theta)$ with θ being the wave normal angle, between the wave vector and some preferred axis of symmetry.

Maxwel's equations

Microscopic	Macroscopic
$\left[\nabla \times \vec{B}\right] = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$	$\left[\nabla \times \vec{H}\right] = \vec{J}_{\mathrm{r}} + \frac{\partial \vec{D}}{\partial t}$
$\left[\nabla \times \vec{E}\right] = -\frac{\partial \vec{B}}{\partial t}$	$\left[\nabla \times \vec{E}\right] = -\frac{\partial \vec{B}}{\partial t}$
$\nabla \cdot \vec{E} = \frac{\rho_q}{\epsilon_0}$	$\nabla \cdot \vec{D} = \rho_{\mathrm{r}}$
$\nabla \cdot \vec{B} = 0$	$\nabla \cdot \vec{B} = 0$

- The charge is divided into a real charge and a polarization charge, $\rho_q = \rho_{\mathrm{r}} + \rho_{\mathrm{p}}$.
- The current is divided into a real current and a magnetization current, $\vec{J} = \vec{J}_{\mathrm{r}} + \vec{J}_{\mathrm{m}}$.
- We will count all charges as polarization charges, thus $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$. All the currents are included in $\frac{\partial \vec{D}}{\partial t}$. We also usually assume that $\vec{B} = \mu_0 \vec{H}$.

Conductivity and dielectric tensors

- We need to specify the relationship between $\vec{D}$ and $\vec{E}$.
- We can solve the equations of motion for the particles and relate $\tilde{\vec{v}}$ with $\tilde{\vec{E}}$. Then

$$\tilde{\vec{J}} = \sum_s n_s e_s \tilde{\vec{v}}_s$$

  can be reorganized to the form

$$\tilde{\vec{J}} = \overleftrightarrow{\sigma} \cdot \tilde{\vec{E}}.$$

- Assuming that $\tilde{\vec{D}} = \epsilon_0 \overleftrightarrow{K} \cdot \tilde{\vec{E}}$ we can use the Maxwell equations and obtain

$$\overleftrightarrow{K} = \overleftrightarrow{1} - \frac{\overleftrightarrow{\sigma}}{\mathrm{i}\omega\epsilon_0}.$$

- We can obtain a homogeneous equation for the electric field:

$$\left[\vec{k} \times \left[\vec{k} \times \tilde{\vec{E}}\right]\right] + \frac{\omega^2}{c^2} \overleftrightarrow{K} \cdot \tilde{\vec{E}} = 0.$$

Waves in a cold uniform unmagnetized plasma

- Assumptions:

$$n_s = n_{s0} + n_{s1}, \quad \vec{v}_s = \vec{v}_{s1}, \quad , \vec{E} = \vec{E}_1, \quad \vec{B} = \vec{B}_1 .$$

- We obtain

$$\overleftrightarrow{\sigma} = \overleftrightarrow{1} \sum_s \frac{n_{s0} e_s^2}{(-\mathrm{i}\omega)\, m_s}, \qquad \overleftrightarrow{K} = \overleftrightarrow{1} \left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2} \right) .$$

  Here $\omega_\mathrm{p}^2 = \sum_s \omega_{\mathrm{p}s}^2 = \sum_s \frac{n_{s0} e^2}{\epsilon_0 m_s}$.

- Assuming $\vec{k} = k\vec{n}_z$, we get

$$\begin{bmatrix} -c^2k^2 + \omega^2 - \omega_p^2 & 0 & 0 \\ 0 & -c^2k^2 + \omega^2 - \omega_p^2 & 0 \\ 0 & 0 & \omega^2 - \omega_p^2 \end{bmatrix} \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0.$$

Transverse and longitudinal modes

- For the transverse mode,

$$\omega^2 = \omega_{\mathrm{p}}^2 + c^2 k^2, \qquad \tilde{\vec{E}} = \left(\tilde{E}_x, \tilde{E}_y, 0\right) .$$

  Waves can exist only for $|\omega| > \omega_{\mathrm{p}}$.

- For the longitudinal mode

$$\omega^2 = \omega_{\mathrm{p}}^2, \qquad \tilde{\vec{E}} = \left(0, 0, \tilde{E}_z\right) .$$

  This mode corresponds to the electron plasma oscillations, which we used to motivate ω_{p}.

Waves in a cold uniform magnetized

- Now we consider a situation with an externally imposed static uniform magnetic field, $\vec{B} = \vec{B}_0 + \vec{B}_1$.
- The situation becomes much more difficult, as the equations couple (we take $\vec{B}$ to point in the z direction):

$$-\mathrm{i}\omega m_s \tilde{v}_{sx} = e_s \left(\tilde{E}_x + \tilde{v}_{sy} B_0 \right),$$

$$-\mathrm{i}\omega m_s \tilde{v}_{sy} = e_s \left(\tilde{E}_y - \tilde{v}_{sx} B_0 \right),$$

$$-\mathrm{i}\omega m_s \tilde{v}_{sz} = e_s \tilde{E}_z.$$

- Expressions for $\overleftrightarrow{\sigma}$ and $\overleftrightarrow{K}$ are complicated.

Waves in a cold uniform magnetized

- Choosing the coordinate system so that $\vec{n} = (n\sin(\theta), 0, n\cos(\theta))$ after some manipulations we can obtain

$$\begin{bmatrix} S - n^2\cos^2(\theta) & -\mathrm{i}D & n^2\sin(\theta)\cos(\theta) \\ \mathrm{i}D & S - n^2 & 0 \\ n^2\sin(\theta)\cos(\theta) & 0 & P - n^2\sin^2(\theta) \end{bmatrix} \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{bmatrix} = 0.$$

Here $S = \frac{1}{2}(R + L)$, $D = \frac{1}{2}(R - L)$, and P being some functions of ω.

Propagation parallel to the magnetic field

- We take $\theta = 0$, thus $\vec{n} = (0, 0, n)$. Then there are the following roots and eigenvectors:

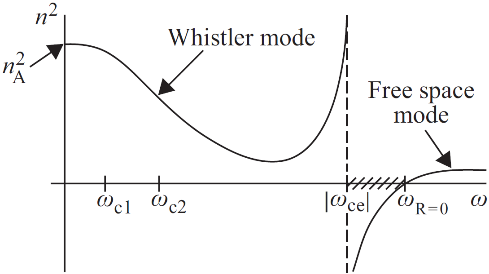
$$P = 0, \qquad \tilde{\vec{E}} = (0, 0, E_0)\,,$$

$$n^2 = R, \qquad \tilde{\vec{E}} = (E_0, \mathrm{i}E_0, 0)\,,$$

$$n^2 = L, \qquad \tilde{\vec{E}} = (E_0, -\mathrm{i}E_0, 0)\,.$$

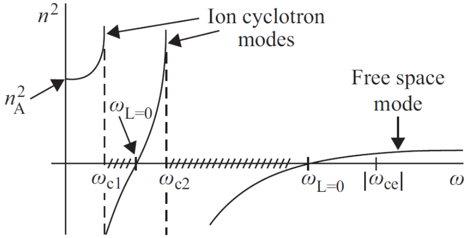
$n^2 = R$ mode

$$n^2 = R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega\,(\omega + \omega_{cs})}$$

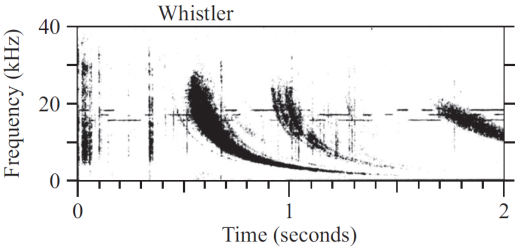


$n^2 = L$ mode

$$n^2 = L = 1 - \sum_s \frac{\omega_{ps}^2}{\omega\,(\omega - \omega_{cs})}$$

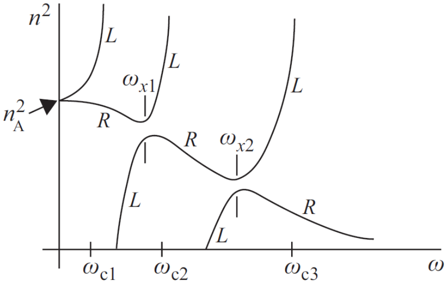


Whistlers



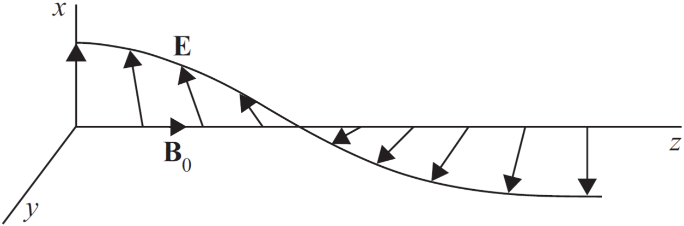
Ion cyclotron whistlers

- Ion cyclotron whistlers exhibit polarization reversal.



Faraday rotation

- For frequencies above $\omega_{R=0}$ both left-hand and right-hand polarized free space modes can propagate. The phase velocities are different for both modes. Due to this, the axis of the electric field polarization rotates as the wave propagates.



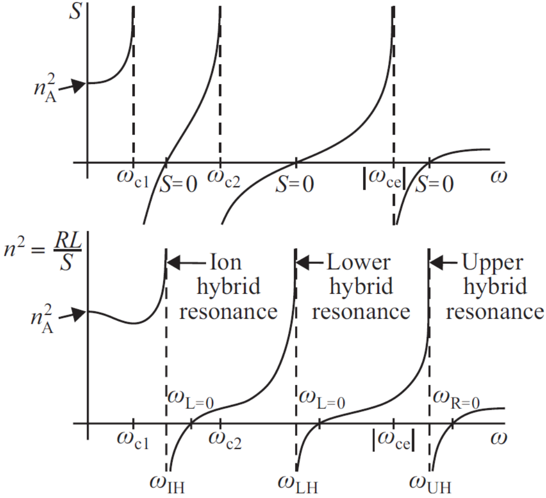
Propagation perpendicular to the magnetic field

- We take $\theta = \pi/2$, thus $\vec{n} = (n, 0, 0)$. Then there are the following roots and eigenvectors:

$$n^2 = P, \qquad \tilde{\vec{E}} = (0, 0, E_0)\,, \qquad \text{ordinary mode.}$$

$$n^2 = \frac{RL}{S}, \qquad \tilde{\vec{E}} = \left(\frac{\mathrm{i}D}{S} E_0, E_0, 0\right), \qquad \text{extraordinary mode.}$$

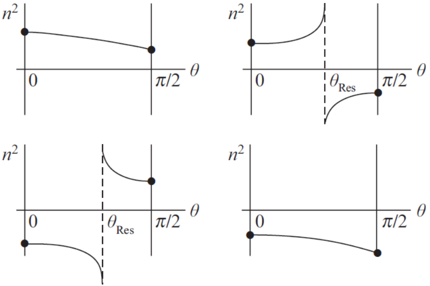
Resonances



S
n_A^2
ω_{c1}
$S=0$
ω_{c2}
$S=0$
$|\omega_{ce}|$
$S=0$
ω
$n^2 = \frac{RL}{S}$
n_A^2
Ion hybrid resonance
Lower hybrid resonance
Upper hybrid resonance
$\omega_{L=0}$
$\omega_{L=0}$
$\omega_{R=0}$
ω_{c1}
ω_{c2}
$|\omega_{ce}|$
ω
ω_{IH}
ω_{LH}
ω_{UH}

Oblique wave propagation

- For $0 < \theta < \pi/2$, the situation is complicated. n goes to infinity at the resonance angle θ_{Res}.



Oblique wave propagation

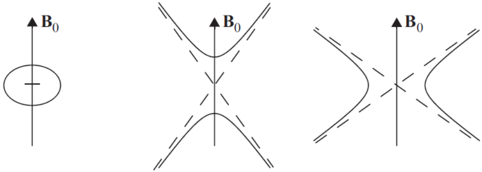


Figure 4.32 The three possible real index of refraction surfaces in a cold magnetized plasma.

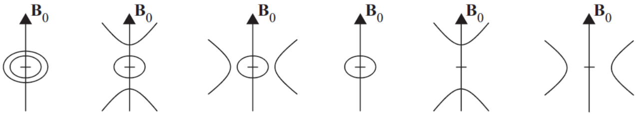
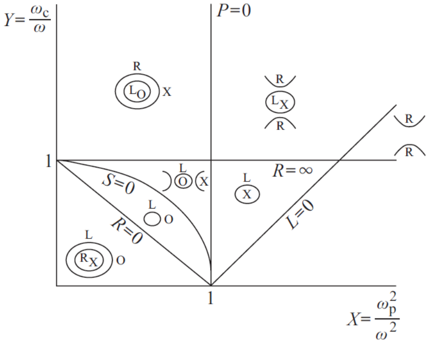


Figure 4.33 The six possible combinations of index of refraction surfaces that can occur for the two electromagnetic modes of propagation in a cold magnetized plasma.

The Clemmow–Mullaly–Allis (CMA) Diagram

- A CMA diagram consists of a diagram with one coordinate for each parameter of the plasma, such as n_s and B_0. Within this parameter space a set of bounding surfaces is constructed defined by the equations $R = 0$, $L = 0$, $P = 0$, $S = 0$, $R = \infty$, and $L = \infty$.



Ray paths in inhomogeneous plasmas

- It is complicated...