Review of Chapters 2 and 3 from *Introduction to Plasma Physics: with Space, Laboratory and Astrophysical Applications* by Gurnett and Bhattacharjee

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2024 09 27

Recap

- We started this book last school year (2023 09 29 to be exact).
- We had 26 seminars, with 16 of them covering the material from the chapters and 10 devoted to the exercises.
- We covered 5 chapters that comprise 185 pages of the book.
- We did 54 exercises.
- All the information about the seminars is in the seminar webpage, in either English (`https://web.vu.lt/ff/a.gelzinis/plasma-physics-seminars/`) or Lithuanian (`https://web.vu.lt/ff/a.gelzinis/plasmos-fizikos-seminarai/`).

Introduction

- A plasma is an ionized gas with approximately equal number of positively and negatively charged particles.
- Plasmas can be produced by ionization, which occurs at high temperatures (thousands of K), thus a plasma could be said to be the "fourth" state of matter.
- Though a plasma is taken to be neutral overall, local deviations from charge neutrality can occur, which induces local electric fields and currents.
- In plasmas the charged particles are usually in an unbound gaseous state, thus their kinetic energy is much larger than the potential energy. In such a plasma, long-range forces are the most important.

Chapter 2: key parameters

- The 2nd Chapter is focused on the key parameters of a plasma.
- Plasmas usually consist of electrons and one or more species of ions, thus the parameters are related to a specific species s.
- The key parameters are the number density n_s, which is the number of particles in a unit volume, and the temperature T_s, which is related to the average kinetic energy by $\left\langle \frac{1}{2} m_s v^2 \right\rangle = \frac{3}{2} k_\mathrm{B} T_s$.
- A very important concept is the velocity distribution function $f_s\left(\vec{v}\right)$. It was discussed more in Chapter 5.
- In thermal equilibrium, the velocities are distributed according to the Maxwellian distribution

$$f_s\left(\vec{v}\right) = n_s \left(\frac{m_s}{2\pi k_\mathrm{B} T_s} \right)^{3/2} \exp\left(-\frac{m_s v^2}{2 k_\mathrm{B} T_s} \right).$$

Chapter 2: Debye length

- There is a fundamental length-scale in plasmas, which characterizes the shielding of test charges.
- A negative test charge Q in a plasma is described by a potential

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}\mathrm{e}^{-r/\lambda_\mathrm{D}}.$$

- Here λ_D is the Debye length, which is given by

$$\lambda_\mathrm{D}^2 = \frac{\epsilon_0 k_\mathrm{B} T_\mathrm{e}}{n_0 e^2},$$

  with n_0 being the ion density.

Chapter 2: plasma frequency

- If the electrons are displaced in a plasma, a restoring force arises. The restoring force is given by Hooke's law.
- Therefore, the system acts like a harmonic oscillator.
- The oscillation frequency is the plasma frequency

$$\omega_{\mathrm{pe}}^2 = \frac{n_0 e^2}{\epsilon_0 m_{\mathrm{e}}}.$$

- For multiple species in a plasma,

$$\omega_{\mathrm{p}} = \sqrt{\sum_s \omega_{\mathrm{p}s}^2}, \qquad \omega_{\mathrm{p}s}^2 = \frac{n_s e_s^2}{\epsilon_0 m_s}.$$

- Note also that

$$\omega_{\mathrm{p}s} \lambda_{\mathrm{D}s} = C_s, \qquad C_s = \sqrt{\frac{k_{\mathrm{B}} T_s}{m_s}}.$$

Chapter 2: cyclotron frequency

- A charged particle in a static magnetic field moves in a circular motion.
- The frequency of this motion is the cyclotron frequency

$$\omega_{\mathrm{c}s} = \frac{e_s B}{m_s}.$$

Chapter 2: collisions

- Collisions provide a means for a plasma to reach equilibrium. Also they tend to damp some phenomena, like oscillations or waves.
- Collision frequency ν_{rs} is the average rate at which r particles collide with s particles.
- Collisions of charged particles with neutral particles are often described using the hard sphere model. Then

$$\nu_{ns} = n_n C_s \sigma_n,$$

  with σ_n being the collision cross section with neutral atoms.
- Collisions between charged particles are more difficult to describe. The key parameter is the scattering cross section $\sigma_C(\chi)$. Here

$$\sigma_C(\chi)\,\mathrm{d}\Omega = \frac{\text{number of particles scattered into } \mathrm{d}\Omega}{\text{incident beam intensity}}$$

Chapter 2: number of electrons per Debye cube

- $N_\mathrm{D} = n_0 \lambda_\mathrm{D}^3$ plays a fundamental role in plasma physics.
- Plasma can be considered continuous only if $N_\mathrm{D} \gg 1$.
- The ratio of the average kinetic energy to the average potential energy can be estimated by

$$\frac{\text{kinetic energy}}{\text{potential energy}} = 6\pi N_\mathrm{D}^{2/3}.$$

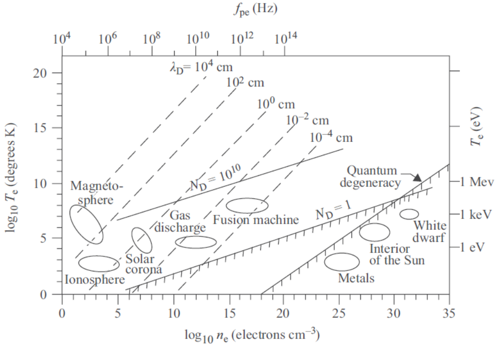
- N_D also related the ratio of collective to discrete interactions, since

$$\frac{\omega_\mathrm{pe}}{\nu_\mathrm{ei}} = \sqrt{\frac{\pi}{2}}\frac{128 N_\mathrm{D}}{\ln\left(12\pi N_\mathrm{D}\right)}.$$

Chapter 2: quantum effects

- Classical description is valid when the product of momentum and position uncertainties is large. This can be written as

$$\frac{(3m_e k_B T_e)^{1/2}}{n_e^{1/2}} \gg \hbar.$$



Chapter 3: motions of a single particle

- Particles move in a configuration of electric and magnetic fields.
- The motion of said particles induces electric and magnetic fields.
- The circle closes due to the self-consistency requirement. The problem becomes difficult.
- For now, we will assume that the fields are fixed. This will simplify the analysis.

Chapter 3: motion in a static uniform magnetic field

- The Newton's second law reads as

$$m\frac{\mathrm{d}}{\mathrm{d}t}\vec{v} = q\left[\vec{v}\times\vec{B}\right].$$

- It is convenient to divide the motion into parallel to the magnetic field, described by $\vec{v}_{\|}$, and into perpendicular to the field, described by $\vec{v}_{\perp}$.
- In such configuration, the particle moves in a circle around the direction of $\vec{B}$. This motion is characterized by the cyclotron radius $\rho_{\mathrm{c}} = \frac{mv_{\perp}}{|q|B}$ and the cyclotron frequency $\omega_{\mathrm{c}} = \frac{v_{\perp}}{\rho_{\mathrm{c}}} = \frac{|q|B}{m}$.

Chapter 3: magnetic moment

- Charged particle moving in a circle has a magnetic moment.
- In general, the magnetic moment is a product of the current and the area of the loop,

$$\mu = IA.$$

- In case of the static magnetic field,

$$\mu = \frac{mv_\perp^2}{2B} = \frac{w_\perp}{B}.$$

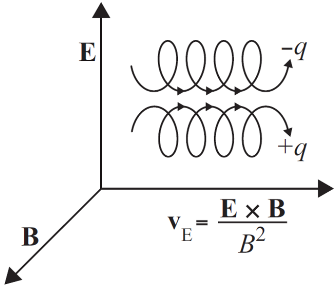
Chapter 3: motion in static and uniform $\vec{E}$ and $\vec{B}$

- The Newton's second law reads as

$$m\frac{\mathrm{d}}{\mathrm{d}t}\vec{v} = q\vec{E} + q\left[\vec{v}\times\vec{B}\right].$$

- There is a new feature in the motion of the particle. It begins to drift with the so-called $\left[\vec{E}\times\vec{B}\right]$ drift velocity

$$\vec{v}_{\mathrm{E}} = \frac{\left[\vec{E}\times\vec{B}\right]}{B^2}.$$

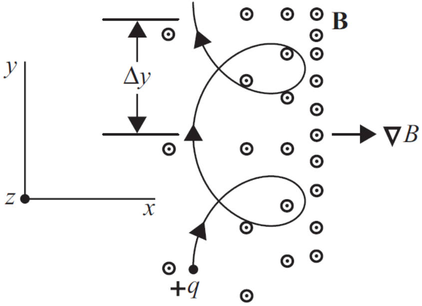


Chapter 3: gradient and curvature drifts

- Gradient drift arises if the magnetic field has a gradient.
- The gradient drift velocity can be expressed as

$$\vec{v}_{\mathrm{G}} = \frac{w_\perp}{qB}\frac{\left[\hat{B}\times\nabla B\right]}{B}.$$

with $w_\perp$ being the perpendicular kinetic energy, $w_\perp = mv_\perp^2/2$.



Chapter 3: gradient and curvature drifts

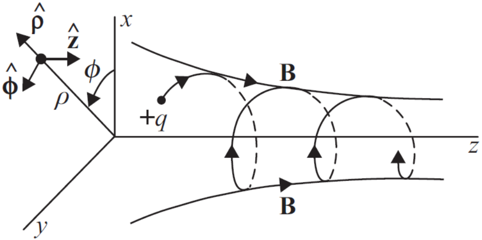
- Curvature drift arises if the magnetic field lines are curved.
- The curvature drift velocity can be expressed as

$$\vec{v}_{\mathrm{C}} = \frac{2w_{\parallel}}{qB} \frac{\left[\hat{B} \times \hat{R}_{\mathrm{C}}\right]}{R_{\mathrm{C}}}.$$

  with $\hat{R}_{\mathrm{C}}$ being a unit vector in the direction of the instantaneous radius of curvature.
- Note that the gradient and curvature drift velocities depend on the charge of the particle, thus they induce a current, whereas the $\left[\vec{E} \times \vec{B}\right]$ drift does not.
- The gradient and and curvature drift affects high energy particles more strongly.
- Also, the gradient and curvature drifts can cause energy gain and losses as the particles drift onto different electrostatic potential contours.

Chapter 3: motion in a magnetic mirror field

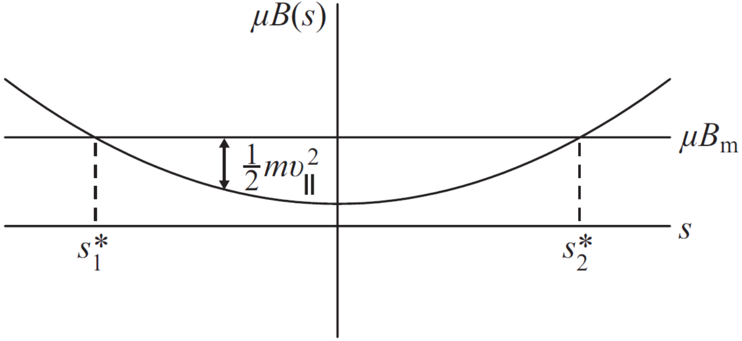
- A magnetic mirror field is characterized by a spatial variation in the magnetic field strength along the magnetic field lines.



Chapter 3: motion in a magnetic mirror field

- It can be shown that the magnetic moment is constant.
- The motion along the field lines can then be described by

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}mv_{\parallel}^2 + \mu B\right) = 0 \quad \Rightarrow \quad \frac{1}{2}mv_{\parallel}^2 + \mu B = \mu B_{\mathrm{m}}$$



Chapter 3: motion in a time varying magnetic field

- If the magnetic field varies in time, from Faraday's law $\left[\nabla \times \vec{E}\right] = -\frac{\partial \vec{B}}{\partial t}$ we see that there is a non-conservative electric field present.
- The total energy of the particle is not conserved.
- It can be shown that the magnetic moment is constant.

Chapter 3: polarization drift

- If the field configuration is time-dependent, an additional drift is produced in the direction of $\frac{\mathrm{d}}{\mathrm{d}t}\vec{E}$.
- The polarization drift velocity can be shown to be

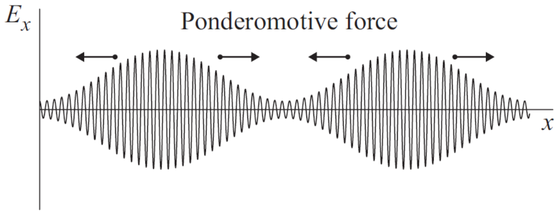
$$\vec{v}_{\mathrm{P}} = \frac{m}{qB^2}\frac{\mathrm{d}\vec{E}}{\mathrm{d}t}.$$

Chapter 3: ponderomotive force

- A charged particle that is placed in an inhomogeneous rapidly oscillating electric field experiences a force that repels it from the region of the strongest field.
- The force can be expressed as

$$\vec{F}_{\mathrm{P}} = -\frac{q^2}{4m\omega^2}\nabla\left(E_0^2\right)$$

with ω being the oscillation frequency of the field.

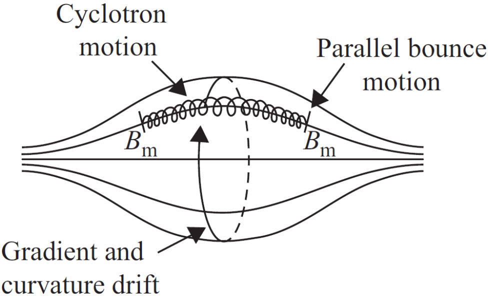


Chapter 3: adiabatic invariants

- In a nearly periodic system with slowly varying parameters, the action integral

$$J = \oint \mathrm{d}q\, p$$

  is an approximate constant of motion.
- An action integral exists for each degree of freedom that exhibits a periodicity.
- For an axially symmetric magnetic mirror we have



Chapter 3: adiabatic invariants

- Adiabatic invariants for an axially symmetric magnetic mirror:
  - the first adiabatic invariant is related to the cyclotron motion and is the magnetic moment
$$\mu = \frac{|q|}{2}\omega_{\mathrm{c}}\rho_{\mathrm{c}}^2.$$
  - the second adiabatic invariant is related to the parallel bounce motion along the magnetic field line and is
$$J = \sqrt{2\mu m}\oint \mathrm{d}s\,\sqrt{B_{\mathrm{m}} - B\,(s)}.$$
  - the third adiabatic invariant is related to the azimuthal gradient and curvature drift around the symmetry axis and corresponds o the magnatic flux
$$\Phi_{\mathrm{B}} = \pi R^2 B.$$

Chapter 3: the Hamiltonian method

- If there is a symmetry in a system, there is a corresponding constant of motions.
- This can be approached using the Hamiltonian method.
- The Hamiltonian is defined as

$$H(q, p, t) = \sum_i \dot{q}_i p - \mathcal{L}(q, \dot{q}, t)$$

  with $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$.
- The Hamiltonian equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}.$$

Chapter 3: the Hamiltonian method

- For a charged particle in an electromagnetic field

$$\mathcal{L} = \frac{m}{2}\dot{\vec{r}}^2 - q\left(\Phi - \vec{A}\cdot\dot{\vec{r}}\right).$$

- The conjugate momentum is

$$\vec{p} = m\dot{\vec{r}} + q\vec{A}.$$

- The Hamiltonian is

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\Phi.$$

- For specific situations we can use a suitable coordinate system and see if H is independent of some q_i or p_j.

Chapter 3: Hamiltonian chaos

- Chaotic trajectories can exist in plasma...