Plasma physics seminar 2023 11 17
Chapter 3
Single particle motion

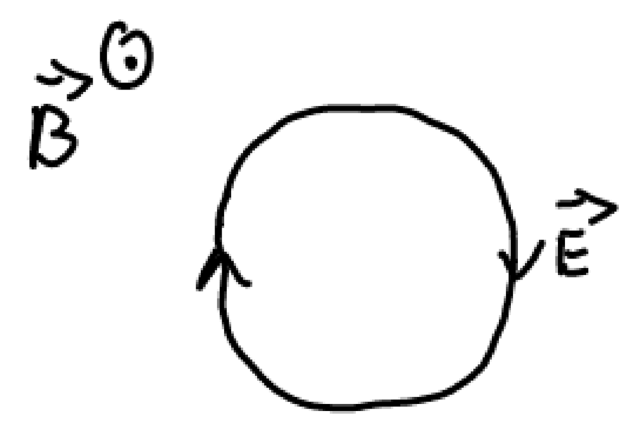
3.5 Motion in a Time Varying Magnetic Field

For a time varying field, Faraday's law applies

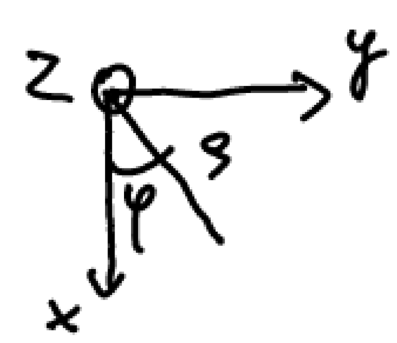
$$\left[\nabla \times \vec{E}\right] = -\frac{\partial \vec{B}}{\partial t}$$

In this case the LHS must also be non zero, thus the electric field cannot be expressed as a gradient of a potential. Thus we have a non-conservative electric field and the total energy of the particle is not constant.

Let us consider a uniform magnetic field that is varying much more slowly than the cyclotron period.



$$\frac{\partial B_z}{\partial t} > 0$$



$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{A} \Rightarrow \oint \vec{E}\cdot d\vec{l} < 0$$

if $d\vec{l}$ is directed counterclockwise? then $\vec{E}$ is directed clockwise.

A charged particle in this fiel gains energy per orbit

$$\Delta W_\perp = q\oint_C \vec{E}\cdot d\vec{l} = -q\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

For $+q$ C is clockwise

for $-q$ C is counterclockwise

Here positive sense of C is in the direction of the particle motion.

$d\vec{A}$ changes direction

Assuming slowly varying fields and $A \approx \pi \varrho_c^2$ we get

$$\Delta W_\perp = |q|\frac{dB}{dt}\cdot \pi \varrho_c^2$$

Cyclotron periodi is $\Delta t = \frac{2\pi}{\omega_c}$.

Thus

$$\frac{dw_\perp}{dt} = \frac{1}{2\pi}\omega_c |q| \frac{dB}{dt}\pi \rho_c^2$$

We had one of the expressions for the magnetic moment

$$\mu = \frac{|q|}{2}\omega_c \rho_c^2$$

Therefore

$$\frac{dw_\perp}{dt} = \mu\frac{dB}{dt} \quad \Rightarrow \quad \mu = \frac{w_\perp}{B} \quad \Rightarrow \quad \frac{dw_\perp}{dt} = \frac{w_\perp}{B}\frac{dB}{dt} \quad \Rightarrow \quad \frac{dw_\perp}{w_\perp} = \frac{dB}{B}$$

$$\Rightarrow \ln w_\perp = \ln B + \ln C \quad \Rightarrow \quad \frac{w_\perp}{B} = C = \text{const}$$

For a time-dependent magnetic field the magnetic moment is constant.

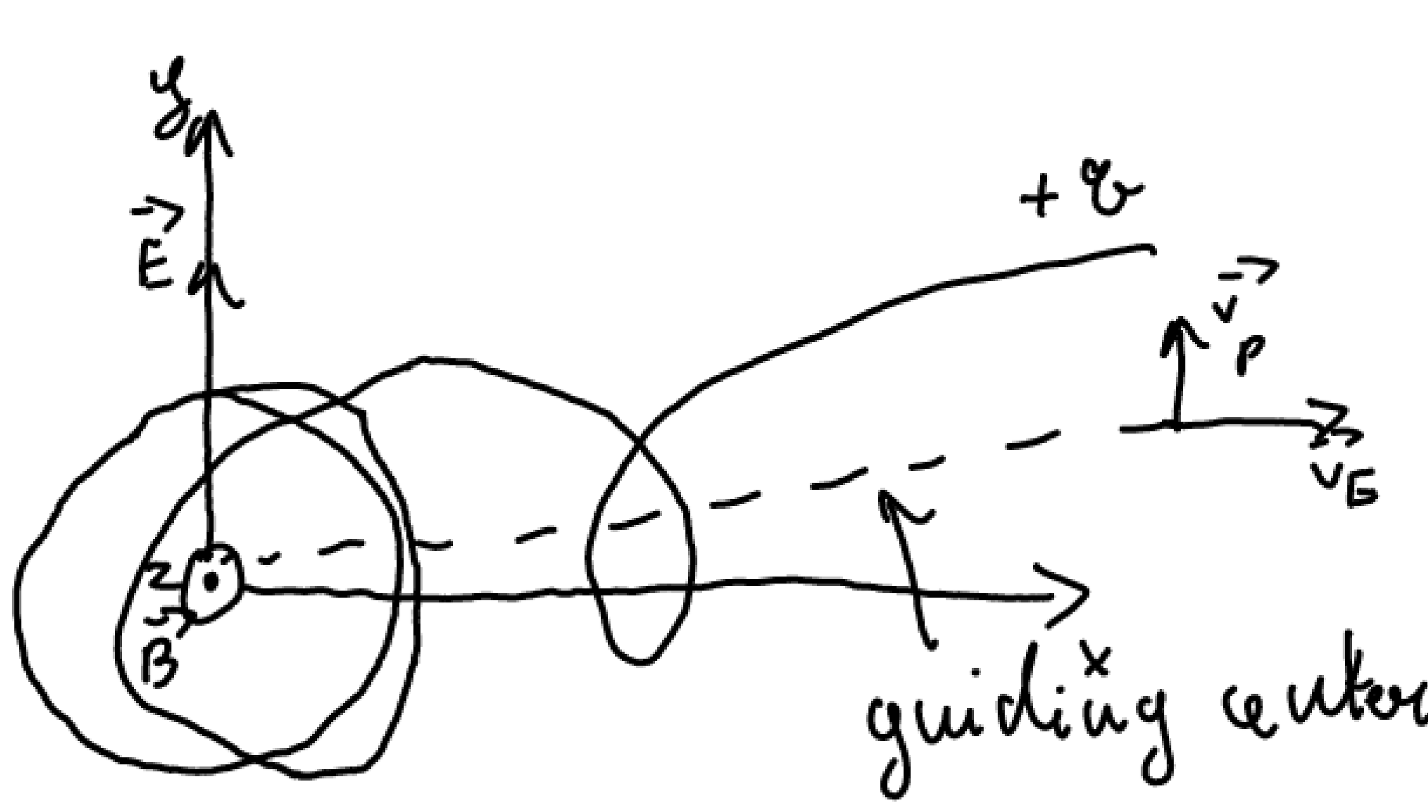
By a change of coordinates the motion in a converging magnetic field can be converted to a time-dependent magnetic field. (In both cases $\mu = \text{const}$)

The conservation of the magnetic moment is equivalent to maintaining a constant magnetic flux through the cyclotron orbit.

$$\mu = \frac{|q|}{2}\omega_c \rho_c^2 = \left[\omega_c = \frac{|q|B}{m}\right] = \frac{q^2}{2m}B\rho_c^2 = \frac{1}{2\pi}\frac{q^2}{m}\Phi_B \quad \text{with } \Phi_B = B\cdot\pi\rho_c^2$$

3.6 Polarization drift

If $\vec{E} = \vec{E}(t)$ in a $[\vec{E}\times\vec{B}]$ field geometry, then there is an additional drift in the direction of $\frac{d\vec{E}}{dt}$. For positive charges this is in the direction of $\frac{d\vec{E}}{dt}$, for negative charges - in the opposite direction. Therefore this drift causes a change of polarization and is thus called the polarization drift.



$E = 0 \ , \ t < 0$

$E = kt \ , \ t > 0$

$\vec{v}_P$ - polarization drift

We will now assume that $\frac{d\vec{E}}{dt}$ is very small.

Two time-scales: 1) rapid time-scale associated with the cyclotron motion; 2) very slow time-scale associated with the electric field and polarization drifts.

We start with the Lorentz force equation

$$m\frac{d\vec{v}}{dt} = q\vec{E} + q\left[\vec{v}\times\vec{B}\right] \qquad (1)$$

We will obtain the drift velocity of the guiding center by averaging the perpendicular velocity, $\langle \vec{v}_\perp \rangle$.

We multiply $(1)\times\frac{\vec{B}}{qB^2}$:

$$\frac{m}{qB^2}\frac{d}{dt}\left[\vec{v}\times\vec{B}\right] = \underbrace{\frac{\left[\vec{E}\times\vec{B}\right]}{B^2}}_{\vec{v}_E} - \frac{1}{B^2}\underbrace{\left[\vec{B}\times\left[\vec{v}\times\vec{B}\right]\right]}_{\vec{v}B^2-\vec{B}(\vec{B}\cdot\vec{v})}$$

The parallel velocity is $\vec{v}_\parallel = \frac{\vec{B}(\vec{v}\cdot\vec{B})}{B^2}$. We also have $\vec{v}-\vec{v}_\parallel = \vec{v}_\perp$. Thus

$$\frac{m}{qB^2}\frac{d}{dt}\left[\vec{v}\times\vec{B}\right] = \vec{v}_E - \vec{v}_\perp \quad \Rightarrow \quad \vec{v}_\perp = \vec{v}_E - \frac{m}{qB^2}\frac{d}{dt}\left[\vec{v}\times\vec{B}\right]$$

We compute the drift velocity by averaging $\langle\vec{v}_\perp\rangle$ over one orbit. Since the cyclotron motion is very fast $\vec{E}$ is constant in the average. Therefore,

$$\vec{v}_d = \langle\vec{v}_\perp\rangle = \vec{v}_E - \frac{m}{qB^2}\cdot\left[\left\langle\frac{d\vec{v}}{dt}\right\rangle\times\vec{B}\right]$$

$$\left\langle\frac{d\vec{v}}{dt}\right\rangle = \frac{q}{m}\vec{E} + \frac{q}{m}\left[\langle\vec{v}\rangle\times\vec{B}\right],$$

This is confusing. Apparently, we should assume

1) $\left\langle\frac{d\vec{v}}{dt}\right\rangle = \frac{d}{dt}\langle\vec{v}\rangle$, not sure about this
2) $\langle\vec{v}\rangle\times\vec{B} = -\vec{E}$, would only make sense if $\left\langle\frac{d\vec{v}}{dt}\right\rangle = 0$

We should obtain

$$\vec{v}_d = \vec{v}_E + \underbrace{\frac{m}{qB^2}\frac{d\vec{E}}{dt}}_{\vec{v}_P}$$

polarization drift, depends on the sign of the charge.

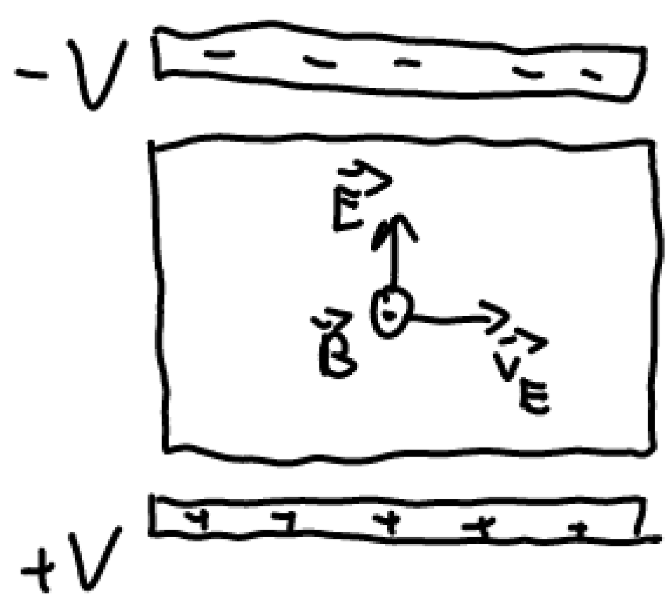
Polarization drift produces a current:

$$\vec{j}_P = \frac{n_0(m_e+m_i)}{B^2}\frac{d\vec{E}}{dt} \quad \Rightarrow \text{dominated by ions.}$$

Now we would like to compare the charge polarization caused by the polarization drift to the charge polarization that develops when an electric field is applied to a solid dielectric material. In the latter case

$$\vec{D} = \epsilon_0 K \vec{E}$$

We consider plasma placed between the plates of a capacitor



We gradually increase the electric field from zero to some final value by changing the voltage.

$\vec{J}_P$ is produced due to the time varying electric field

A surface charge density $\sigma_P = \int dt\, \vec{J}_P \cdot \hat{n}$ develops.

Apparently

$$\vec{P} \cdot \hat{n} = \sigma_P \quad \Rightarrow \quad \int dt\, \vec{J}_P \cdot \hat{n} = \int dt\, \frac{n_0(m_e + m_i)}{B^2} \frac{d\vec{E}}{dt} \cdot \hat{n} = \frac{n_0(m_e + m_i)}{B^2} \vec{E} \cdot \hat{n}$$

Therefore

$$\vec{P} = \epsilon_0 \underbrace{\left[\frac{n_0(m_i + m_e)}{\epsilon_0 B^2} \right]}_{\chi \text{ - dielectric susceptability}} \vec{E}$$

Finally,

$$K = 1 + \chi = 1 + \frac{n_0(m_i + m_e)}{\epsilon_0 B^2}$$

We should later see that this dielectric constant accounts for some specific waves in a plasma.