Plasma physics seminar 2023 10 20

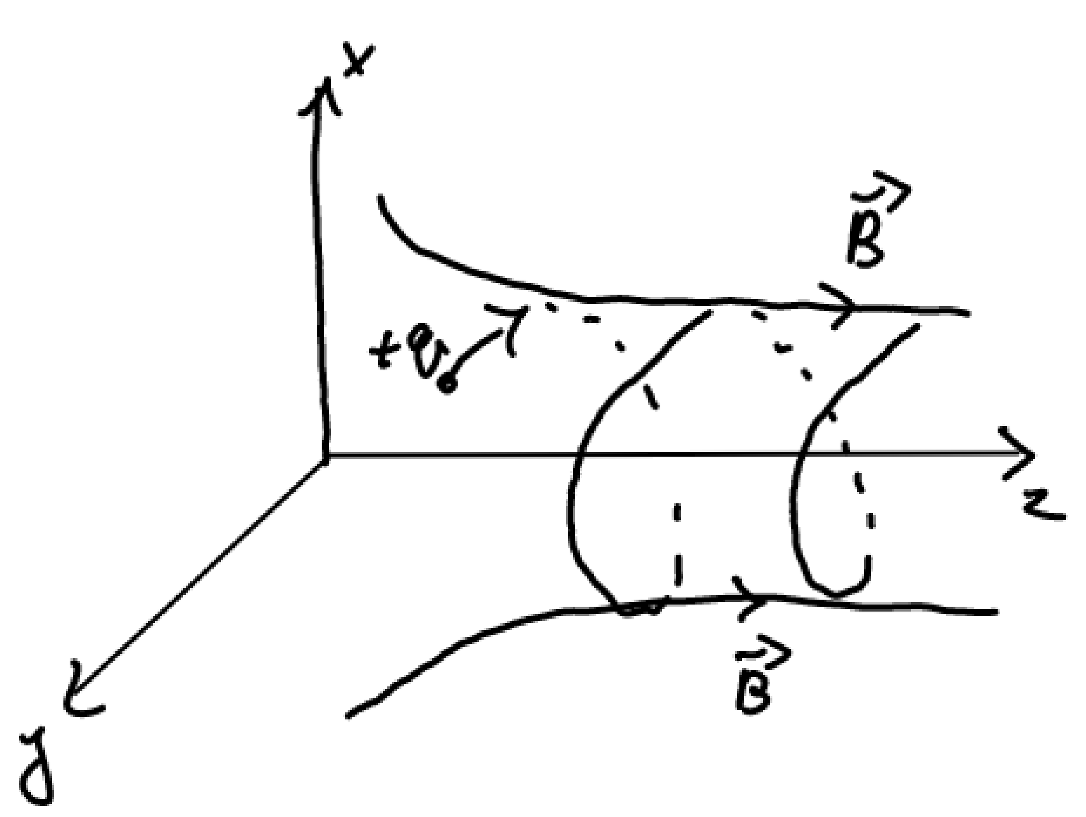
Chapter 3

Single particle motions

3.4 Motion in a Magnetic Mirror field

A magnetic mirror field is such that there is a spatial variation in the magnetic field strength along the magnetic field lines.
It is said that this variation leads to a reflection of the parallel component of the guiding center motion.

We will assume an axially symmetric field and use the cylindrical coordinates.



We have Maxwell's equation $\nabla \cdot \vec{B} = 0$

The field lines should converge.

$$\nabla \cdot \vec{B} = \frac{1}{\varrho}\frac{\partial}{\partial \varrho}(\varrho B_\varrho) + \frac{\partial B_z}{\partial z} = 0$$

this should be valid near the symmetry axis

$$\Rightarrow \frac{\partial}{\partial \varrho}(\varrho B_\varrho) = -\varrho \frac{\partial B_z}{\partial z} \quad \Rightarrow \text{we assume that } \frac{\partial B_z}{\partial z} \text{ does not depend on } \varrho$$

$$\Rightarrow \varrho B_\varrho = -\frac{1}{2}\varrho^2 \frac{\partial B_z}{\partial z} + C \quad \Rightarrow B_\varrho = -\frac{1}{2}\varrho\frac{\partial B_z}{\partial z} + \frac{C}{\varrho}$$

To get finite values of B_ϱ at $\varrho = 0$ we have to set $C = 0$.

Since $B_x = B_\varrho \cos\varphi$ and $B_y = B_\varrho \sin\varphi$, we can multiply the previous result by $\cos\varphi$ or $\sin\varphi$ and obtain

$$B_x = -\frac{1}{2}x\frac{\partial B_z}{\partial z}, \qquad B_y = -\frac{1}{2}y\frac{\partial B_z}{\partial z}$$

If $\frac{\partial B_z}{\partial z} > 0$, then $B_\varrho < 0$ and the field lines converge towards the symmetry axis as z increases.

3.4.1 Parallel Motion: The Magnetic Mirror Force

From the Lorentz force we get

$$m \frac{d}{dt}\vec{v} = q\left[\vec{v} \times \vec{B}\right] \quad \Rightarrow \quad m\frac{dv_z}{dt} = q\left[v_x B_y - v_y B_x\right]$$

the equation for parallel motion along the z axis

Using the previous results, we can write the force as

$$F_z = -\frac{q}{2}\left(\frac{\partial B_z}{\partial z}\right)\left(v_x y - v_y x\right)$$

For very slowly varying magnetic fields, the transverse motion is nearly circular, thus

$$x = \rho_c \sin(\omega_c t), \qquad y = \frac{q}{|q|}\rho_c \cos(\omega_c t) \quad \Bigg| \quad \omega_c = \frac{|q|B}{m}, \quad \rho_c = \frac{m v_\perp}{|q|B} = \frac{v_\perp}{\omega_c}$$

$$v_x = \omega_c \rho_c \cos(\omega_c t), \qquad v_y = -\frac{q}{|q|}\omega_c \rho_c \sin(\omega_c t)$$

Then

$$F_z = -\frac{q}{2}\frac{\partial B_z}{\partial z}\cdot\left(\frac{q}{|q|}\omega_c \rho_c^2 \cos^2(\omega_c t) + \frac{q}{|q|}\omega_c \rho_c^2 \sin^2(\omega_c t)\right) =$$

$$= -\frac{|q|}{2}\omega_c \rho_c^2 \frac{\partial B_z}{\partial z}$$

We had the expression for the magnetic moment

$$\mu = I \cdot A \quad \Rightarrow \quad \mu = \frac{m v_\perp^2}{2B} = \frac{m\rho_c^2\omega_c^2}{2B} = \frac{|q|}{2}\omega_c \rho_c^2$$

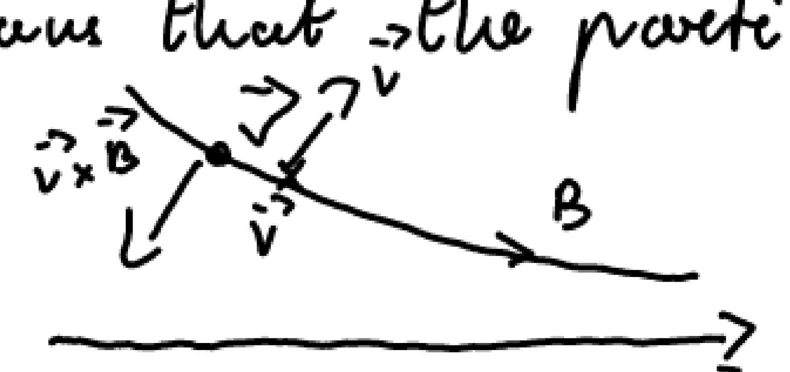
Therefore

$$F_z = -\mu \frac{\partial B_z}{\partial z}$$

Writing the vector magnetic moment as $\vec{m} = -\mu\hat{B}$, we get back the apparently well-known expression for the force on a magnetic moment in an inhomogeneous field:

$$\vec{F} = (\vec{m}\cdot\nabla)\vec{B}$$

The direction of the force means that the particle is repelled from the region of the strong magnetic field.

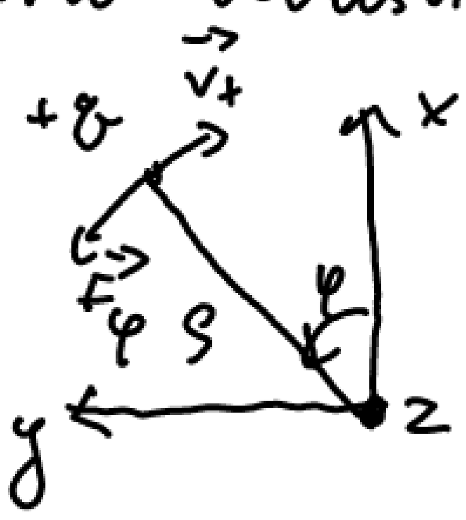


3.4.2 Azimuthal Motion: Constancy of the Magnetic Moment

The Lorentz force has a component in the azimuthal direction

$$\begin{vmatrix} \vec{n}_\rho & \vec{n}_\varphi & \vec{n}_z \\ v_\rho & v_\varphi & v_z \\ B_\rho & 0 & B_z \end{vmatrix} \quad \Rightarrow \quad F_\varphi = q v_z B_\rho$$

An illustration:



[It seems to me that $\vec{F}_\varphi$ should be directed in the opposite direction...]

It is said that $v_\varphi = -\frac{q}{|q|} v_\perp$

The perpendicular kinetic energy changes due to the azimuthal force:

$$\frac{dW_\perp}{dt} = v_\varphi F_\varphi = q v_\varphi v_z B_\rho$$

$$\Rightarrow \frac{dW_\perp}{dt} = q\left(-\frac{q}{|q|} v_\perp\right) v_z \left(-\frac{1}{2}\rho \frac{\partial B_z}{\partial z}\right) = |q| v_\perp v_z \frac{\partial B_z}{\partial z} \frac{\rho}{2}$$

Note that the total kinetic energy remains constant, since $[\vec{v} \times \vec{B}] \perp \vec{v}$.

Now we change $\rho_c = \frac{m v_\perp}{|q| B}$ for ρ:

$$\frac{dW_\perp}{dt} = |q| v_\perp v_z \frac{\partial B_z}{\partial z} \frac{m v_\perp}{2|q| B} = \frac{W_\perp v_z}{B} \frac{\partial B_z}{\partial z}$$

From this we should show that μ is a constant of motion.

$$\frac{d\mu}{dt} = \frac{d}{dt}\frac{W_\perp}{B} = \frac{1}{B}\frac{dW_\perp}{dt} - \frac{W_\perp}{B^2}\frac{dB}{dt};$$

We have $\frac{dB}{dt} = v_\rho \frac{\partial B}{\partial \rho} + v_z \frac{\partial B}{\partial z}$; for nearly circular motion $v_\rho = 0$;

we also hold that only B_z component depends on z. Thus

$$\frac{d\mu}{dt} = \frac{W_\perp v_z}{B^2}\frac{\partial B_z}{\partial z} - \frac{W_\perp}{B^2} v_z \frac{\partial B_z}{\partial z} = 0 \quad \Rightarrow \quad \mu = \text{const.}$$

This is valid only if the magnetic field gradient is small enough.

3.4.3 Turning Points and the Pitch Angle

We have derived the equation for parallel motion along the z axis

$$m \frac{dv_z}{dt} = -\mu \frac{\partial B_z}{\partial z}$$

This is valid for a small local region.

It is said that this equation can be generalized by changing

$z \to s$, $\quad$ s being the distance along the field line;

$v_z \to v_\parallel$,

$B_z \to B$

We thus obtain

$$m \frac{dv_\parallel}{dt} = -\mu \frac{\partial B}{\partial s}$$ / should probably be $\frac{dB}{ds}$

We multiply both sides by $v_\parallel$

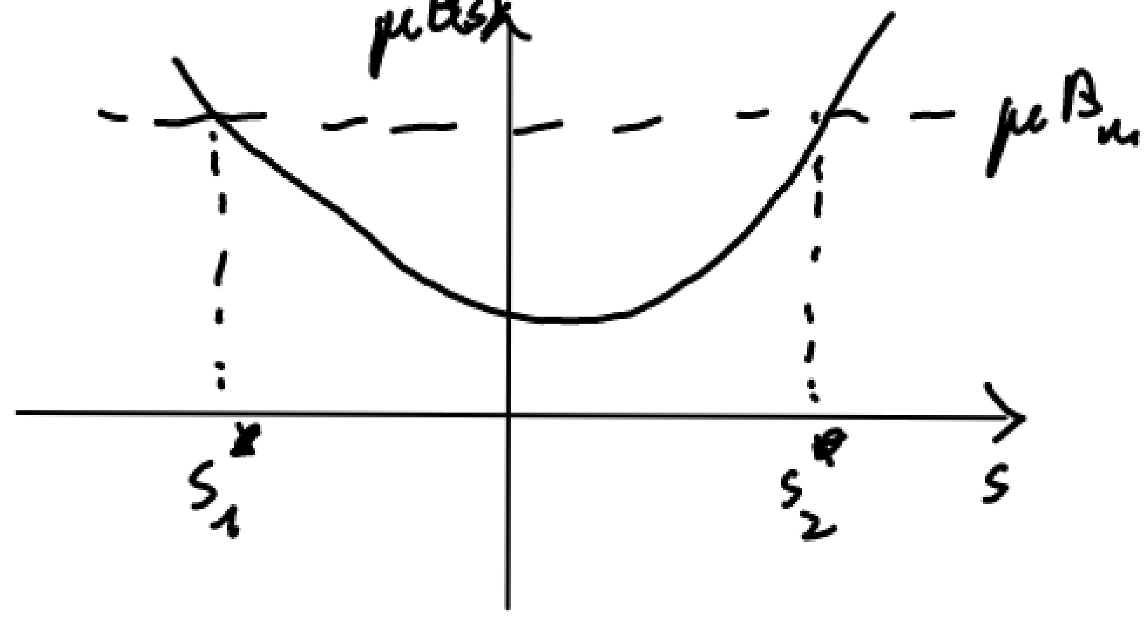
$$m v_\parallel \frac{dv_\parallel}{dt} = -\mu v_\parallel \frac{dB}{ds} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{m v_\parallel^2}{2} + \mu B\right) = 0 \quad // \; v_\parallel = \frac{ds}{dt}$$

Therefore, we obtain an energy conservation law

$$\frac{m v_\parallel^2}{2} + \mu B = \mu B_m = w, \text{ total energy;}$$

This corresponds to a particle of mass m moving in a one-dimensional effective potential $\mu B(s)$.

Typical potential for a mirror magnetic field is



Magnetic bottle configuration

s_1^* and s_2^* are turning points

The motion is determined by

$$v_\parallel = \frac{ds}{dt} = \pm \sqrt{\frac{2\mu}{m}\left(B_m - B\right)}$$

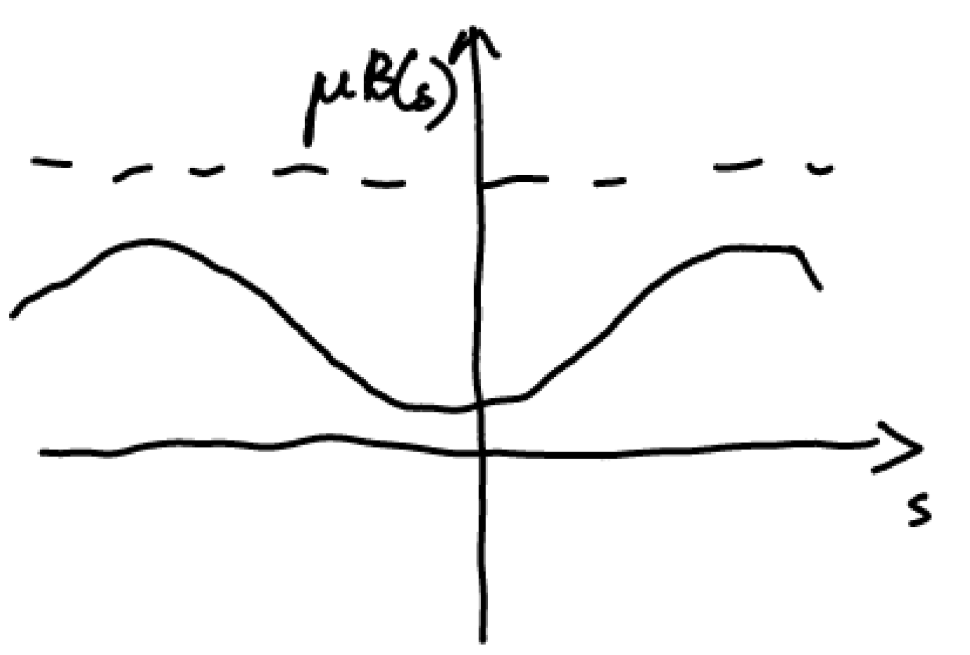
Sometimes we want to know the pitch angle - the angle between the velocity vector and the magnetic field.

Since $v_\perp = v \sin\alpha$, then $W_\perp = W \sin^2\alpha$. From previous results we have

$$\mu = \frac{W_\perp}{B} = \frac{W \sin^2\alpha}{B} \Rightarrow \sin^2\alpha = \frac{W_\perp}{W} = \frac{\mu B}{\mu B_m} = \frac{B}{B_m}$$

If the constant B_m is known, α can be computed at any point s.

The magnetic field cannot be infinitely strong, thus the potential will look like this:

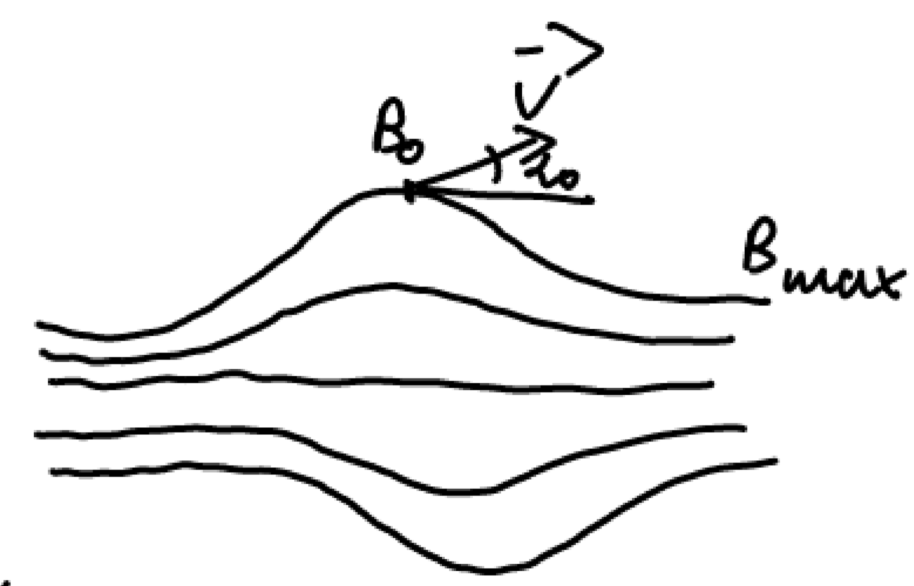


Particles with large kinetic energies can escape the magnetic bottle.

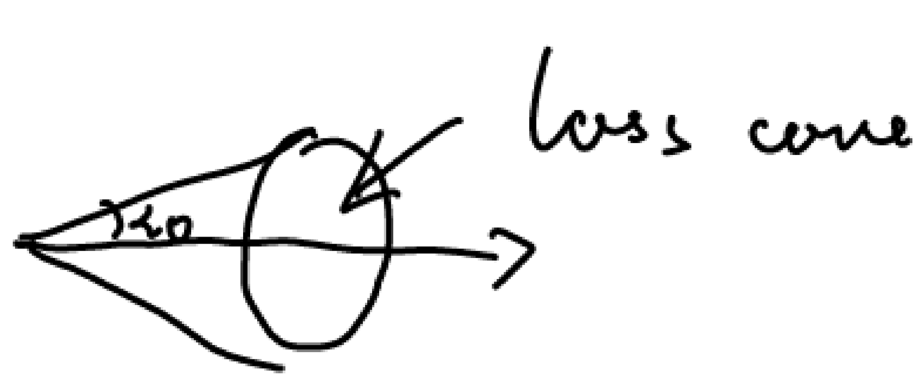
Large kinetic energies mean small pitch angles.

The minimum pitch angle is defined by

$$\sin^2\alpha_0 = \frac{B_0}{B_{max}}$$



Any particle with a pitch angle less than α_0 will escape from the system.



It is said that the loss cone leads to instabilities that affects confinement of plasma in a magnetic mirror machine.

Particles in a planetary radiation belt also have a loss cone

Particles moving with a pitch angle less than α_0 will hit the planet and be lost.