Plasma physics seminar 2023 10 20
Chapter 3
Single particle motions

3.3 Gradient and Curvature Drifts

If the magnetic field is not spatially constant, the guiding center tends to move perpendicularly to the field.
Two types of drifts occur: gradient drifts and curvature drifts.

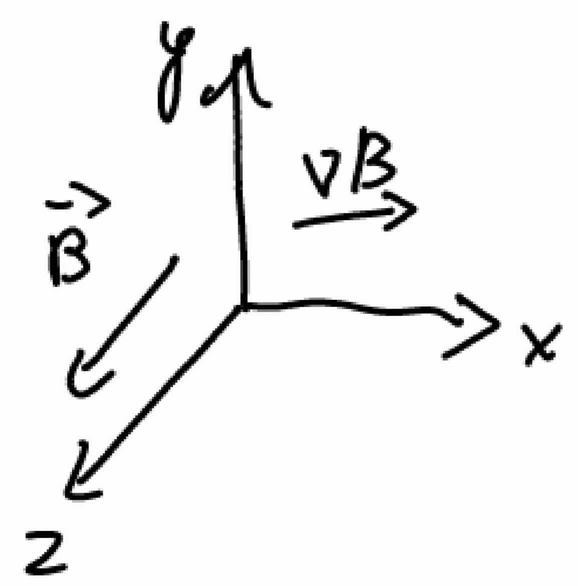
A mathematically elegant approach would be to use the guiding center model and to expand the magnetic field

$$\vec{B}(\vec{r}) = \vec{B}(\vec{r}_0) + (\vec{\rho}_c \cdot \nabla)\vec{B}\Big|_{\vec{r}=\vec{r}_0} + \dots$$

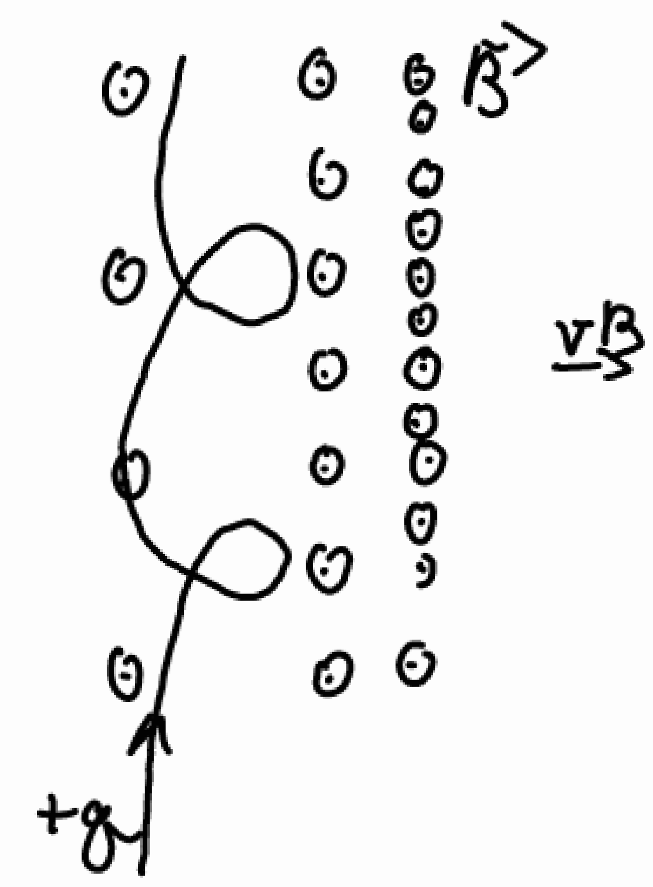
Here $\vec{r}_0$ represents the instantaneous guiding center, and $\vec{\rho}_c$ is the cyclotron radius vector to the magnetic line at $\vec{r}_0$.
The perturbation theory should converge if $\frac{\rho_c}{B}|\nabla B| \ll 1$, but we will not pursue this.
Instead we will analyze simple geometries and try to generalize the results.

3.3.1 Gradient Drift

We consider the following geometry



$\Rightarrow$



We had $\rho_c = \frac{m v_\perp}{|q| B}$

ρ_c decreases at stronger fields and increases at smaller fields

There is cycloid-like drift in y direction.

If the magnetic field variation over the cyclotron orbit is small, a simple description of the orbit is obtained.

The motion in the x direction is periodic

$$m\ddot{\vec{r}} = q\left[\dot{\vec{r}} \times \vec{B}\right] \Rightarrow m\ddot{x} = q\dot{y}B_z(\vec{r})$$
$$m\ddot{y} = -q\dot{x}B_z(\vec{r})$$

$\Rightarrow$ $\cancel{m(\dot{y}-\dot{y}_0) = -q(x-x_0)B_z(\vec{r})}$

$\cancel{\dot{y} = \dot{y}_0 - \frac{q}{m}B_z(\vec{r})}$

$\cancel{\ddot{x} = -\frac{q^2 B_z}{m^2}x + \frac{q\dot{y}_0 B_z}{m}}$

0

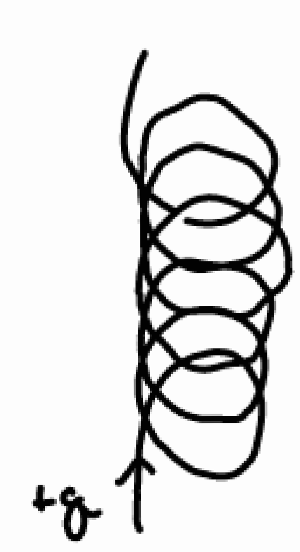
If B_z variation is very small, then we have the equation of harmonic oscillator. Is it true for larger B_z variations? For small velocities, it should be the case.

This implies that

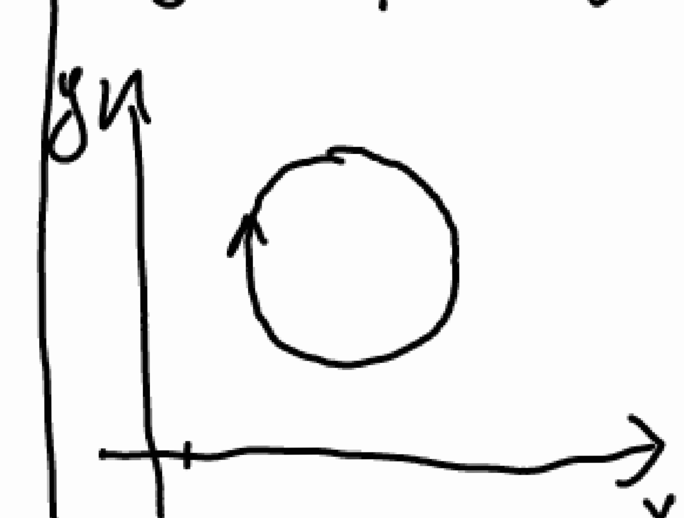
$$\oint dt\, F_x = q\oint dt\, v_y B_z = 0 \quad \Big| \text{ time average over one orbit.}$$

Expanding B_z in a Taylor series

$$B_z(x_0)\underbrace{\oint dt\, v_y}_{\Delta y} + \frac{\partial B_z}{\partial x}\Big|_{x_0}\underbrace{\oint dt\, v_y(x-x_0)}_{=\oint dy\,(x-x_0)\,\approx} = 0$$



$$\oint = \int_{\uparrow} + \int_{\downarrow}$$

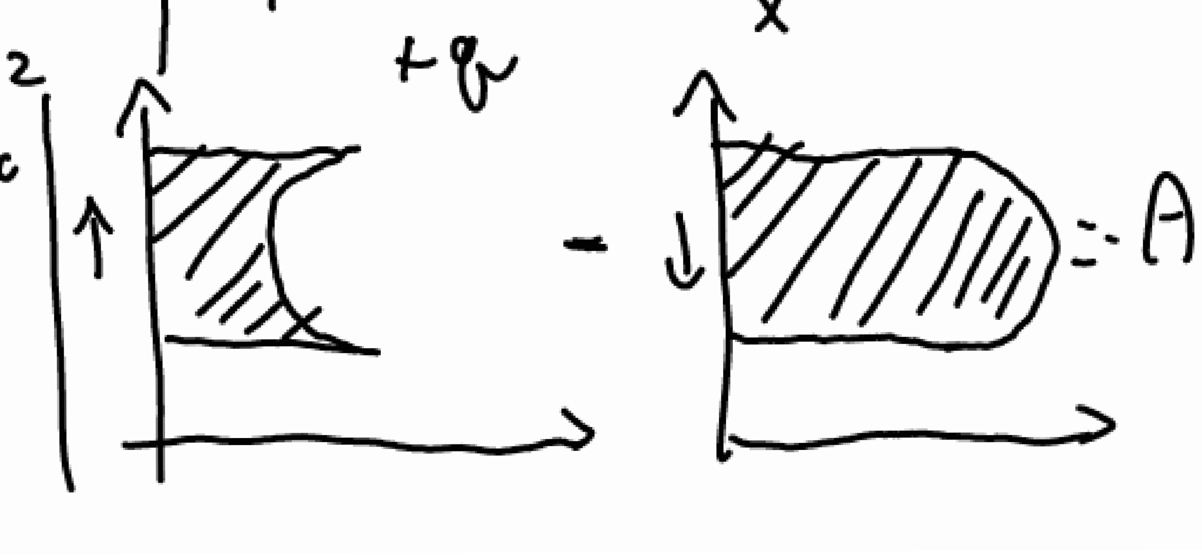


$$\approx -\frac{q}{|q|}\pi\rho_c^2$$

OK

$\omega_c = \frac{|q|B}{m}$

$$\text{Thus, } v_G^{(x_0)} = \frac{\Delta y}{\Delta t} = \frac{1}{\Delta t}\frac{1}{B_z(x_0)}\frac{\partial B_z}{\partial x}\Big|_{x_0}\frac{q}{|q|}\pi\rho_c^2 \,; \quad \Delta t = \frac{2\pi}{\omega_c},\; w_\perp = \frac{m}{2}\omega_c^2\rho_c^2$$



$$\Rightarrow \quad v_G = \frac{\omega_c}{2\cancel{\pi}} \frac{1}{B_z(x_0)} \left.\frac{\partial B_z}{\partial x}\right|_{x_0} \frac{q}{|q|} \cancel{\pi} \rho_c^2 = \frac{1}{m\omega_c} w_\perp \frac{1}{B_z(x_0)} \left.\frac{\partial B_z}{\partial x}\right|_{x_0} \frac{q}{|q|} = \left[m\omega_c = |q| B \right] =$$

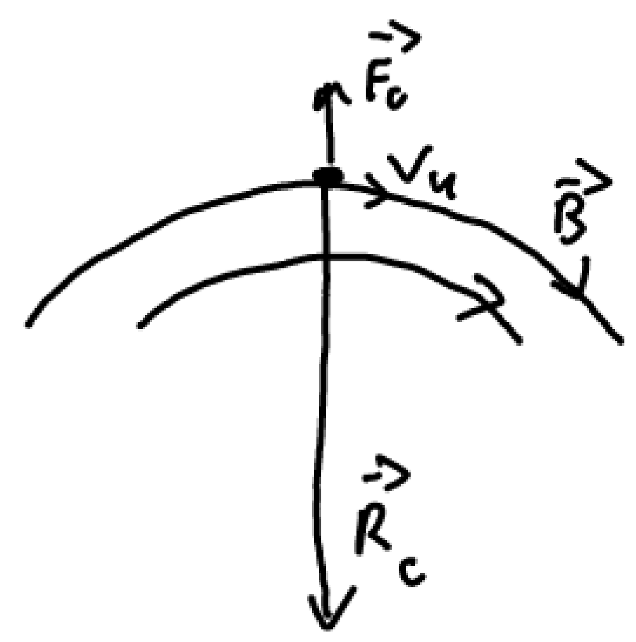
$$= \left.\left(\frac{w_\perp}{q B_z} \frac{1}{B_z} \frac{\partial B_z}{\partial x} \right)\right|_{x_0}$$

Since v_G is along the y axis, we can rewrite this expression to a vector form:

$$\vec{v}_G = \left.\left(\frac{w_\perp}{qB} \underbrace{\frac{[\hat{B} \times \nabla B]}{B}} \right)\right|_{\vec{r}_0}$$

↳ this has units of inverse length and characterizes the gradient of B.

3.3.2 Curvature Drifts



The particle should experience a centrifugal force

$$F_c = \frac{m v_\parallel^2}{R_c} \quad [\text{not straightforward}]$$

Then

$$\vec{v}_c = \frac{\vec{F} \times \vec{B}}{qB^2} = -m \frac{v_\parallel^2}{R_c^2} \frac{[\vec{R}_c \times \vec{B}]}{qB^2} \quad \Big| \quad \vec{F}_c \text{ is directed opposite to } \vec{R}_c$$

Using $w_\parallel = \frac{1}{2} m v_\parallel^2$, we can rewrite this expression as

$$\vec{v}_c = \frac{2 w_\parallel}{qB} \frac{[\hat{B} \times \hat{R}_c]}{R_c}$$

This is similar to the gradient drift.

Both gradient and curvature drifts are proportional to an inverse length-scale that characterizes the inhomogeneity of the magnetic field.

The gradient and curvature drifts tend to have similar magnitudes.

In both cases the drift velocities are proportional to the kinetic energy and inversely proportional to the magnetic field strength.

The magnitude of the drift velocity is controlled by $\frac{\rho_c}{L_{char}}$.

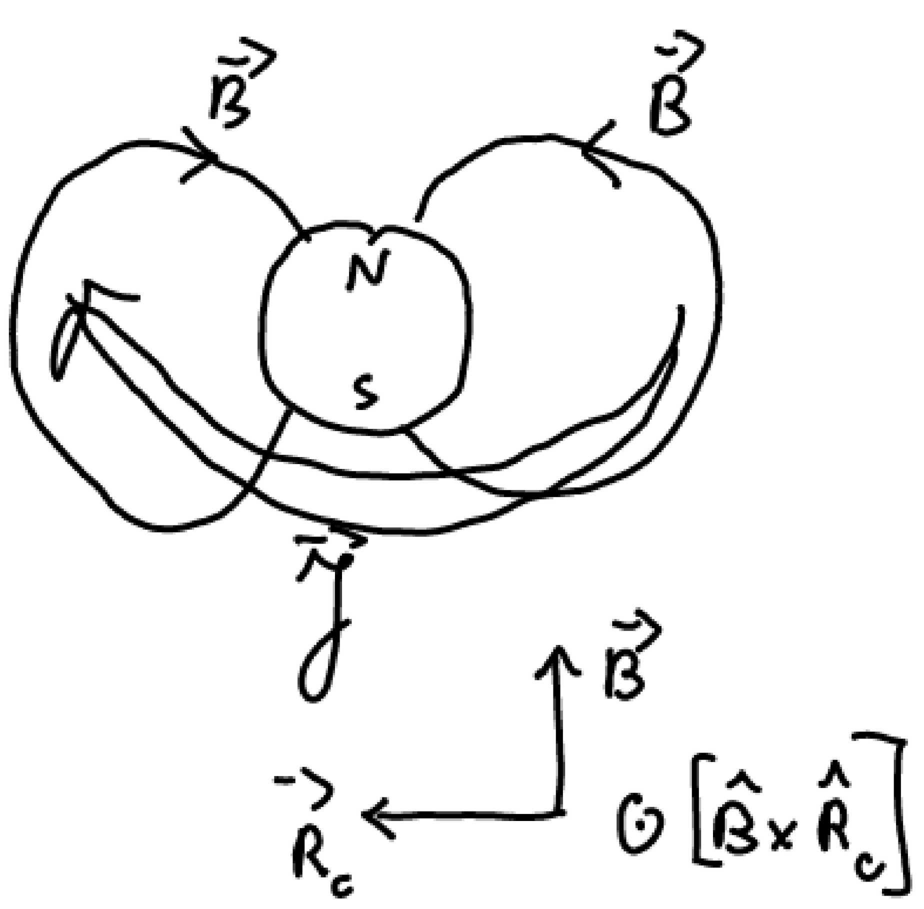
[this is not easy to see for curvature drifts]

Differences from the $[\vec{E}\times\vec{B}]$ drifts:

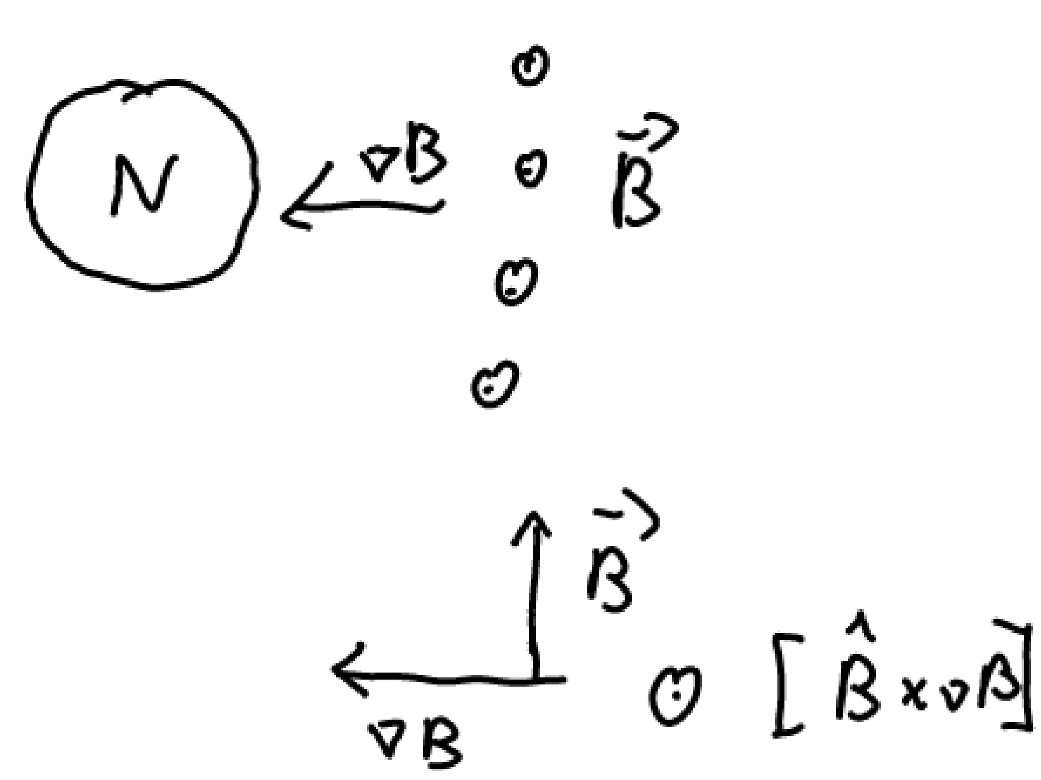
1. $v_G, v_c \sim W$, v_E is independent of W
2. v_G, v_c depend on the sign of q. Gradient and curvature drifts give rise to currents.
3. Gradient and curvature drifts do not necessarily occur along contours of constant electrostatic potential. Particles can thus gain energy.

3.3.3 Examples of Gradient and Curvature Drifts

Earth's radiation belt



For curvature drift

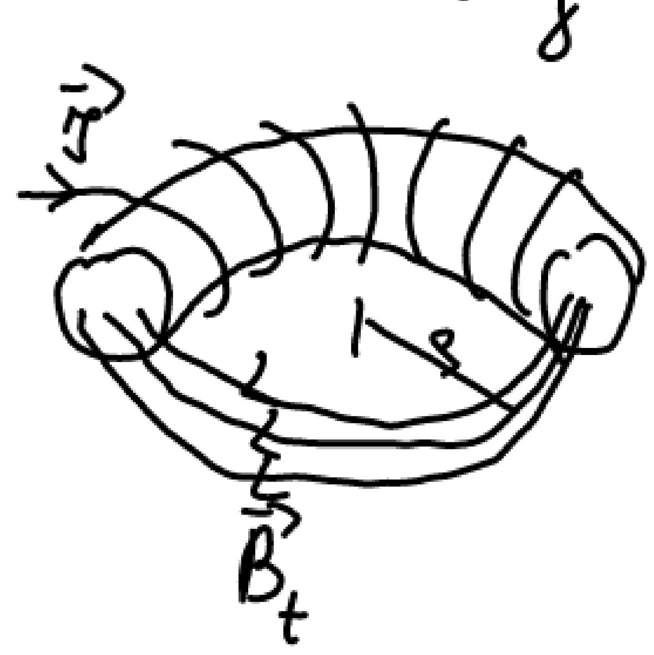


For gradient drift

Curvature and gradient drifts point to the same direction

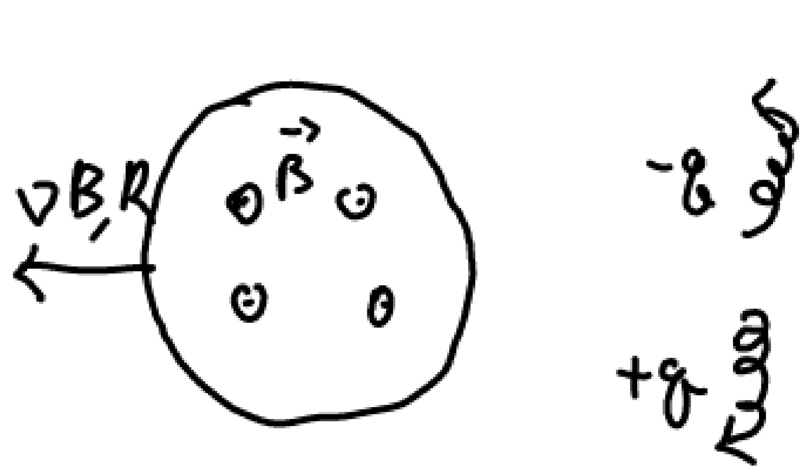
Here $\vec{J}$ is the ring current

Toroidal magnetic field



$$B_t = \frac{\mu_0 N_e}{2\pi}\frac{I}{\varrho}$$ Ampere's law

Radial gradient exists



=> polarization charge builds =>

$$\vec{V}_E = \frac{\vec{E}\times\vec{B}}{B^2}$$

plasma hits the walls

It is possible to introduce an additional poloidal magnetic field to make the field lines no longer close on themselves.

3.3.4 A Self-consistent Static Equilibrium

Now we will discuss a self-consistent plasma configuration in a simple two-dimensional system in which spatially varying magnetic field lines are parallel to the z axis, with $\vec{B} = [0, 0, B_z(x,y)]$.

For plasma in such a field, there will be a magnetization current (density), $\vec{J}_m$, and a gradient drift current, $\vec{J}_G$.

Thus,

$$\vec{M} = -\sum_s n_s \langle \mu_s \rangle \hat{B} = -\hat{z} \sum_s n_s \frac{\langle w_{\perp s} \rangle}{B} = -\hat{z}\frac{W_\perp}{B}$$

Here we defined the average perpendicular kinetic energy density, $W_\perp = \sum_s n_s \langle w_{\perp s} \rangle$

The magnetization current density then is

$$\vec{J}_m = [\nabla \times \vec{M}] = \left[\nabla \times \left(-\hat{z}\frac{W_\perp}{B}\right)\right] = \left[\left(\nabla \frac{W_\perp}{B}\right) \times (-\hat{z})\right] + \frac{W_\perp}{B}\underbrace{[\nabla \times (-\hat{z})]}_{=0} =$$

$$= \left[\hat{z} \times \nabla \frac{W_\perp}{B}\right] = \left[\frac{\hat{z}}{B} \times \nabla W_\perp\right] - \left[\frac{\hat{z} W_\perp}{B^2} \times \nabla B\right] \qquad \left| \; [\nabla \times \varphi \vec{F}] = [\nabla \varphi \times \vec{F}] + \varphi [\nabla \times \vec{F}] \right.$$

The gradient drift current density is

$$\vec{J}_G = \sum_s n_s e_s \langle \vec{v}_{Gs} \rangle = \sum_s n_s \cancel{e_s} \frac{\langle w_{\perp s} \rangle}{\cancel{e_s} B^2} [\hat{z} \times \nabla B] = \frac{W_\perp}{B^2}[\hat{z} \times \nabla B]$$

The total current density is

$$\vec{J} = \vec{J}_m + \vec{J}_G = \left[\frac{\hat{z}}{B} \times \nabla W_\perp\right] - \cancel{\left[\frac{\hat{z} W_\perp}{B^2} \times \nabla B\right] + \frac{W_\perp}{B^2}[\hat{z} \times \nabla B]}$$

To have a self-consistent solution, this current must be related to the magnetic field via the Ampere's law, thus

$$[\nabla \times \vec{B}] = \mu_0 \vec{J} = \frac{\mu_0}{B}[\hat{z} \times \nabla W_\perp]$$

$$\hookrightarrow [\nabla \times \hat{z} B] = -[\hat{z} \times \nabla B]$$

Thus

$$\left[\hat{z} \times \left(\nabla B + \frac{\mu_0}{B}\nabla W_\perp\right)\right] = 0 \;\Big|\; \cdot \frac{B}{\mu_0} \Rightarrow \left[\hat{z} \times \nabla\left(\frac{B^2}{2\mu_0} + W_\perp\right)\right] = 0$$

→ if this is const., eq. is satisfied.

a special case of magnetostatic equilibrium condition:

magnetic field pressure + particle pressure = const.