Plasma physics seminar 2023 10 13
Chapter 3
Single particle motion

To have a complete mathematical model of a plasma, we need to know:

1) motion of all particles for a specific field configuration;
2) charge and current densities resulting from said motions;
3) electric and magnetic fields resulting from said densities.

Since the fields influence the motion of the particles, the circle closes – we have a self-consistency requirement. This makes analysis of plasma difficult.

At first, we will ignore the self-consistency requirement and we will focus on the motion of a single particle in a known field configuration.

This is useful in some situations:

1) The external fields are much stronger than the internal ones;
2) The actual fields are known from measurements;
3) It might be possible to "use the general solution for the particle motion in an assumed field geometry to determine a fully self-consistent solution in which the currents and charges produce the assumed fields." [somewhat unclear]

3.1 Motion in a Static Uniform Magnetic Field

The relevant equation of motion is

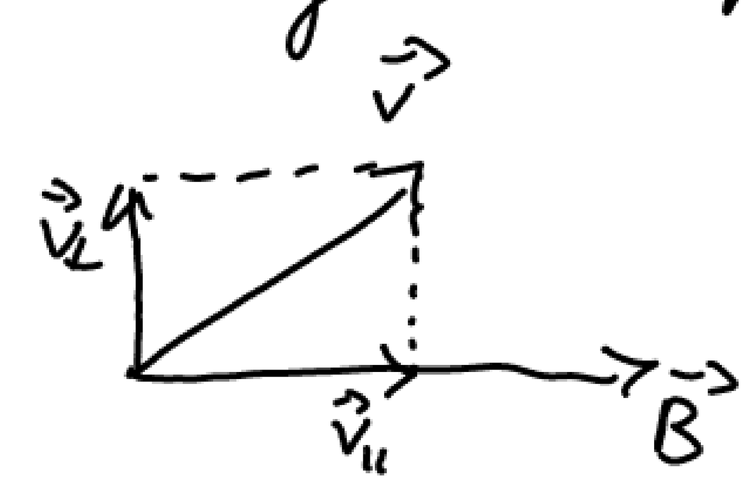
$$m \frac{d\vec{v}}{dt} = q\left[\vec{v} \times \vec{B}\right]$$

$$\Rightarrow m\vec{v} \cdot \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{1}{2} m v^2\right) = q\vec{v} \cdot \left[\vec{v} \times \vec{B}\right] = 0$$

Kinetic energy $w = \frac{1}{2} m v^2$ is a constant of motion.

We will now resolve the velocity to components paralle and perpendicular to the magnetic field:

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$$



We also divide the kinetic energy in such a way: $w_{\parallel/\perp} = \frac{1}{2} m v_{\parallel/\perp}^2$

The Lorentz force is perpendicular to $\vec{B}$, thus $v_{\parallel}$ is constant.

Since w and $w_{\parallel}$ are constants, $w_{\perp}$ and $v_{\perp}$ are also constants.

The particle is moving with constant speed along the magnetic field.

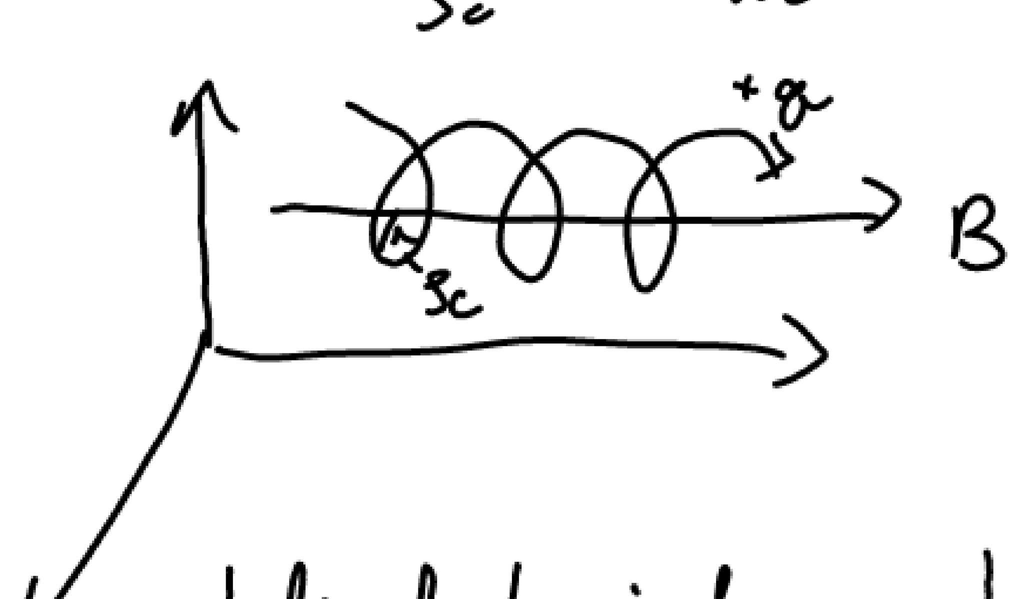
In the plane perpendicular to the field, we have

$$m \frac{d\vec{v}_{\perp}}{dt} = q\left[\vec{v}_{\perp} \times \vec{B}\right] = q v_{\perp} B \left[\vec{n}_{\perp} \times \vec{n}_{\parallel}\right]$$

The speed is constant and the acceleration is perpendicular to the velocity and also of constant magnitude - we have circular motion:

$$m \frac{v_{\perp}^2}{\rho_c} = |q| v_{\perp} B \quad \Rightarrow \quad \rho_c = \frac{m v_{\perp}}{|q| B} \text{ is called the cyclotron radius.}$$

$$\omega_c = \frac{v_{\perp}}{\rho_c} = \frac{|q| B}{m} \text{ is the cyclotron frequency.}$$



Positive particles rotate in the left-hand sense w.r.t $\vec{B}$

Negative - in the right-hand sense w.r.t. $\vec{B}$

helical trajectory

The instantaneous center of the rotational motion is called the guiding center.

3.1.1 Magnetic Moment

The magnetic moment of the current loop is

$$\mu = I A$$

The magnitude of the current

$$I = \frac{|q|}{T_c} = \frac{|q|\omega_c}{2\pi} = \frac{q^2 B}{2\pi m} \quad , \; T_c \text{ - cyclotron period}$$

Area of the current loop

$$A = \pi \rho_c^2 = \pi \left(\frac{m v_\perp}{|q| B} \right)^2$$

Combining everything

$$\mu = \frac{\cancel{q^2} \cancel{B}}{2 \cancel{\pi} \cancel{m}} \cdot \cancel{\pi} \frac{\cancel{m^2} v_\perp^2}{\cancel{q^2} B^{\cancel{2}}} = \frac{m v_\perp^2}{2B} = \frac{W_\perp}{B}$$

The direction of the magnetic moment is described by the right-hand rule. For both positive and negative charges the magnetic moment is oriented opposite to the magnetic field.

A magnetized plasma is always diamagnetic.

The magnetization, $\vec{M}$, is defined as the magnetic moment per unit volume.

In plasma

$$\vec{M} = -\sum_s n_s \langle \mu_s \rangle \hat{B}$$

The magnetization current is given by $\vec{J}_m = [\nabla \times \vec{M}]$

3.2 Motion in Static and Uniform Electric and Magnetic fields

Now we consider a case when in addition to the magnetic field, there is also an electric field, which, for simplicity, we will assume to be perpendicular to the magnetic field. The equation for motion is

$$m\frac{d\vec v}{dt} = q\vec E + q\left[\vec v \times \vec B\right] \qquad // \text{ fields are constant}$$

We again divide the velocity into two terms $\vec v = \vec v_{\parallel} + \vec v_{\perp}$ $\left(\vec B = B\vec n_{\parallel}\right)$

Then

$$m\frac{d\vec v_{\parallel}}{dt} + \frac{m\, d\vec v_{\perp}}{dt} = \overbrace{q\vec E}^{\perp \vec B} + \overbrace{q\left[\vec v_{\perp} \times \vec B\right]}^{\perp \vec B}$$

Since $\vec E \perp \vec B$, the equation reduces to two equations

$$m\frac{dv_{\parallel}}{dt} = 0$$

$$m\frac{d\vec v_{\perp}}{dt} = q\vec E + q\left[\vec v_{\perp} \times \vec B\right]$$

Assuming non-relativistic velocities, we can make a coordinate transformation and eliminate $\vec E$ from the equations.

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Relativistic transformations going to a frame moving with velocity $\vec v$:

$$\vec E' = \gamma\left(\vec E + \left[\vec v \times \vec B\right]\right) - (\gamma - 1)\left(\vec E \cdot \hat v\right)\hat v \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\vec B' = \gamma\left(\vec B - \frac{\left[\vec v \times \vec E\right]}{c^2}\right) - (\gamma - 1)\left(\vec B \cdot \hat v\right)\hat v$$

For small velocities we can take $c \to \infty$, then $\gamma \to 1$ and we obtain

$$\vec E' = \vec E + \left[\vec v \times \vec B\right]$$

$$\vec B' = \vec B$$

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We take the transformation velocity $\vec v_E$ to be perpendicular to both $\vec B$ and $\vec E$. Then

$$v_{\parallel}' = v_{\parallel}, \qquad \vec v_{\perp}' = -\vec v_E + \vec v_{\perp}$$

Then

$$m \frac{dv_{\parallel}'}{dt} = 0$$

$$m \frac{dv_{\perp}'}{dt} = q\left(\vec{E} + [\vec{v}_E \times \vec{B}]\right) + [\vec{v}_{\perp}' \times \vec{B}]$$

For small velocities, we can choose $\vec{v}_E$ to be such that

$$\vec{E}' = \vec{E} + [\vec{v}_E \times \vec{B}] = 0$$

$$\Rightarrow [\vec{B} \times \vec{E}] + [\vec{B} \times [\vec{v}_E \times \vec{B}]] = 0 \quad \Rightarrow [\vec{B} \times \vec{E}] = -\left(\vec{v}_E B^2 - \vec{B}\overset{0}{(\vec{v}_E \cdot \vec{B})}\right)$$

$$\Rightarrow \vec{v}_E = \frac{[\vec{E} \times \vec{B}]}{B^2}$$ $\leftarrow$ this is called the $\vec{E} \times \vec{B}$ drift velocity.

The equation of motion in such moving frame of reference becomes

$$m \frac{d\vec{v}'}{dt} = q[\vec{v}' \times \vec{B}]$$

This describes the motion of a particle in a static uniform magnetic field.

In the original frame of reference, the particle moves in a circular motion around the magnetic field plus with a uniform translational motion in the $\vec{E} \times \vec{B}$ direction with the velocity $\vec{v}_E$.

In case $v_{\parallel} = 0$, the trajectories are cycloids in the $\vec{E}, \vec{v}_E$ plane, with positive charges rotating in the left hand sense with respect to the magnetic field, and negative charges rotating in the right-hand sense.

If $v_{\parallel} \neq 0$, the guiding center, in addition to moving with the drift velocity $\vec{v}_E$, continues to move with $v_{\parallel}$ along the magnetic field.

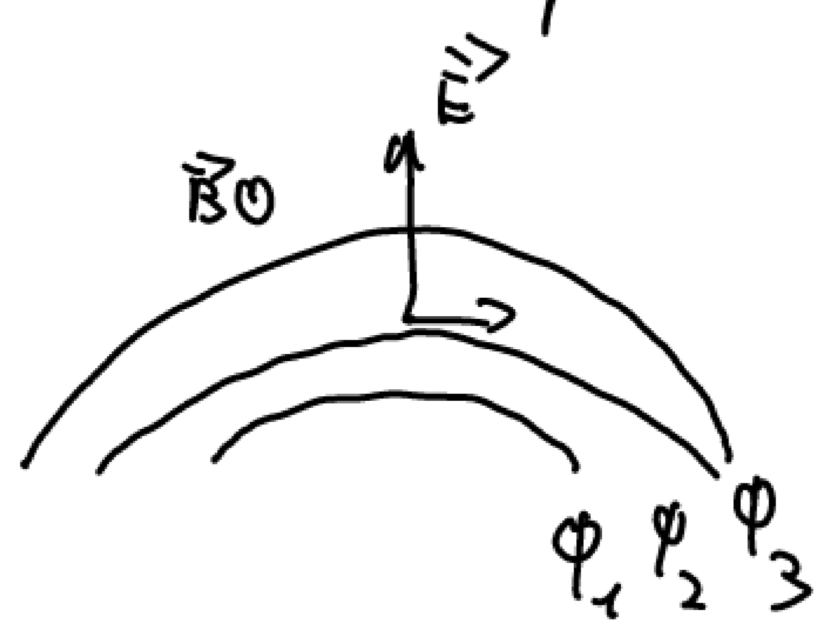
This analysis becomes invalid for strong electric fields, if $\vec{v}_E$ becomes comparable with the field of light.

The features of a particle moving in $\vec{B} + \vec{E}$ fields:

- All particles drift with the same velocity irrespective of their charge, mass, or energy. The electric field is determined by the frame of references and there is no electric field in a frame moving with the drift velocity. The rest frame of the plasma is usually defined as a frame, where $\vec{E}=0$, but for inhomogeneous plasmas this definition applies only locally.
- Since we assume electrically neutral plasmas, the $\vec{E}\times\vec{B}$ drift produces no net current density:

$$\vec{j} = \sum_s e_s n_s \vec{v}_{E,s} = \vec{v}_{E,s} \sum_s e_s n_s = 0$$

- The $\vec{E}\times\vec{B}$ drift velocity is always along a contour of constant electrostatic potential.



The total energy of the particle is not changed by the $\vec{E}\times\vec{B}$ drift.

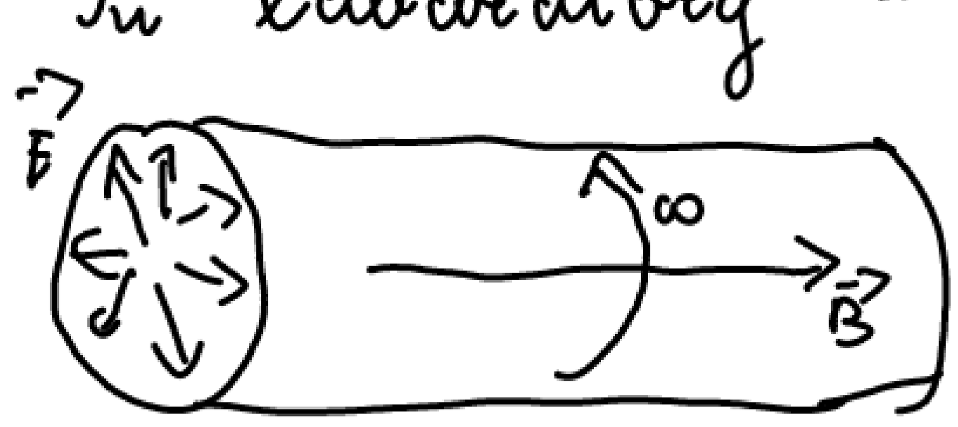
3.2.1 Examples of $\vec{E}\times\vec{B}$ Drifts

1) Rotation of plasma with the planet, co-rotational electric field

$$\vec{E} = -[\vec{v}_E \times \vec{B}] = -[[\vec{\omega}\times\vec{r}]\times B]$$

The region where co-rotation dominates is called the plasmasphere, and the outer boundary of the plasmasphere is called the plasmapause.

2) In laboratory - cylinder with axial magnetic field and radial electric field



Forced rotation

3) In both space and laboratory plasmas, the relationship between the perpendicular electric fields and plasma drifts provide a method to measure the flow velocity. We need to measure the perpendicular electric field and then use $\vec{v}_E = \frac{[\vec{E} \times \vec{B}]}{B^2}$ to compute the flow velocity.

In laboratory plasmas, a conducting probe (Langmuir probe) can be used to measure the potential.

In space plasmas two probes are used to measure the potential

$$E = -\frac{\varphi_2 - \varphi_1}{L}$$

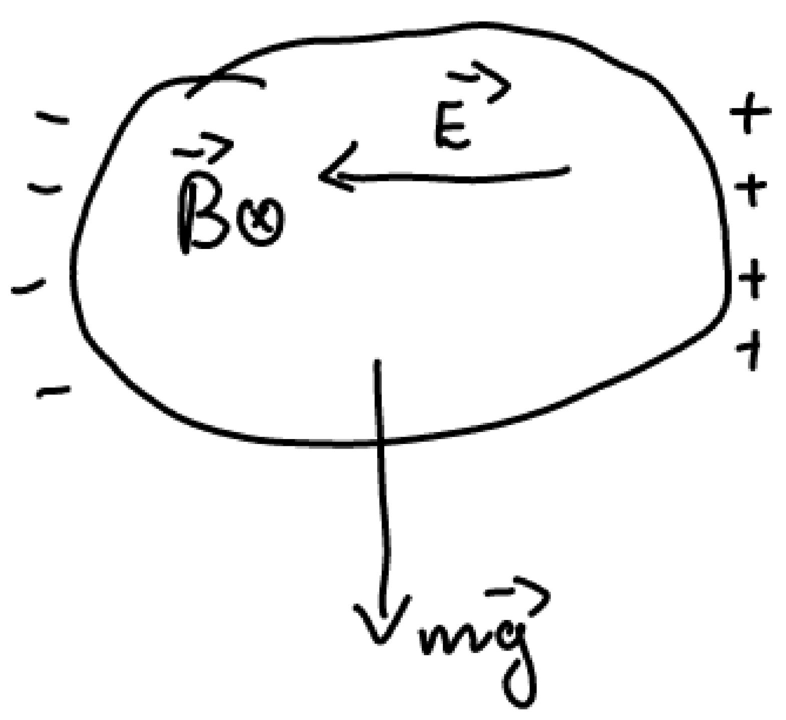
For large separations (100 - 200 m), small fields, down to $0.1 \frac{mV}{m}$ can be measured.

3.2.2. Drift due to a Force Perpendicular to $\vec{B}$

The $\vec{E} \times \vec{B}$ drift can be generalized to include drifts due to external forces that are perpendicular to the magnetic field. Defining $\vec{E}_\perp = \frac{\vec{F}_\perp}{q}$, we get

$$\vec{V}_F = \frac{[\vec{F}_\perp \times \vec{B}]}{q B^2}$$

An example - the gravitational force $m\vec{g}$.



Drift does not cause the plasma to fall directly.

Indirectly, due to the drift polarization field develops.

This leads to downwards drift.

It is possible to inhibit the polarization electric field.

In most laboratory and space plasmas, the effect of the gravitational force is much smaller than that of the electric and magnetic forces.