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Cosmology

Chapter 3. Thermal History

(The first 3 minutes in the history of the Universe)

1/17/2023

The Hot Big Bang

- Rate of interactions $\Gamma \gg$ Rate of expansion H
  - Timescale of particle interactions $\ll$ characteristic expansion time scale
  - Local thermal equilibrium is reached before effect of expansion becomes relevant

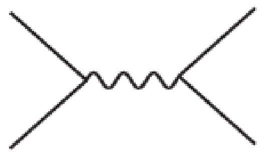
- Natural units: $c = \hbar = k_{\mathrm{B}} = 1$, Planck mass $M_{\mathrm{Pl}} = \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \cdot 10^{18}\ \mathrm{GeV}$

- Rate of interactions $\Gamma = n\sigma v$
  - Velocity of particles $v \approx 1$

$\boxed{E = \sqrt{m^2 + p^2} \approx p}$

  - Number density of particles $n = \int f(p,T)\,\mathrm{d}^3p \sim \int \mathrm{e}^{-E/T}\ 4\pi p^2\,\mathrm{dp} \propto \int \mathrm{e}^{-p/T}\,p^2\,\mathrm{dp}$

$\propto T^3$

  - Interaction cross-section $\sigma \sim \left| \right.$  $\left. \right|^2 \propto \frac{\alpha^2}{T^2}$

$\alpha = \frac{g_A^2}{4\pi} \sim 0.01$ – structure constant associated with the gauge boson A

$$\Gamma \propto T^3 \cdot \frac{\alpha^2}{T^2} \propto \alpha^2 T$$

Local thermal equilibrium

$\Gamma \gg H$

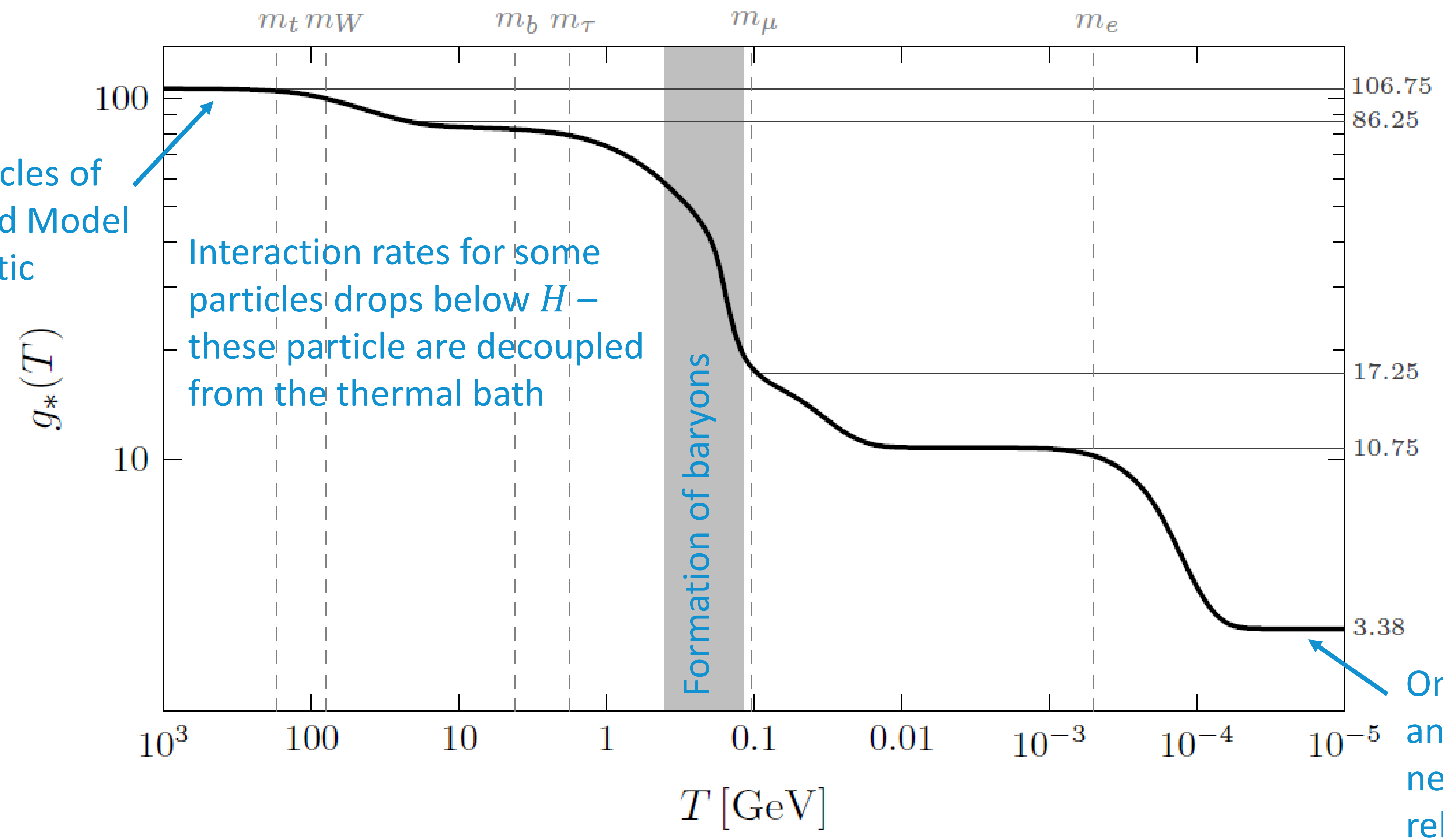
- Rate of interactions $\Gamma = n\sigma v \propto \alpha^2 T$
- Hubble rate $H \sim \sqrt{\rho}/M_{\mathrm{Pl}}$
  - Density $\rho = \int f(p,T)\, E\, \mathrm{d}^3 p \;\propto\; \int \mathrm{e}^{-p/T} \cdot p \cdot p^2\, \mathrm{d}p \;\propto T^4$

$$\hookrightarrow \quad H \propto \frac{T^2}{M_{\mathrm{Pl}}} \quad \Rightarrow \quad \frac{\Gamma}{H} \propto \frac{\alpha^2 M_{\mathrm{Pl}}}{T} \propto \frac{10^{16}\ \mathrm{GeV}}{T}$$

- Thus for temperatures below $T \sim 10^{15}$ GeV and above 100 GeV (relativistic limit) all the particles of the Standard model are in thermal equilibrium
- As T drops below the mass of the particles, they become non-relativistic, yielding $f \propto \mathrm{e}^{-m/T}$

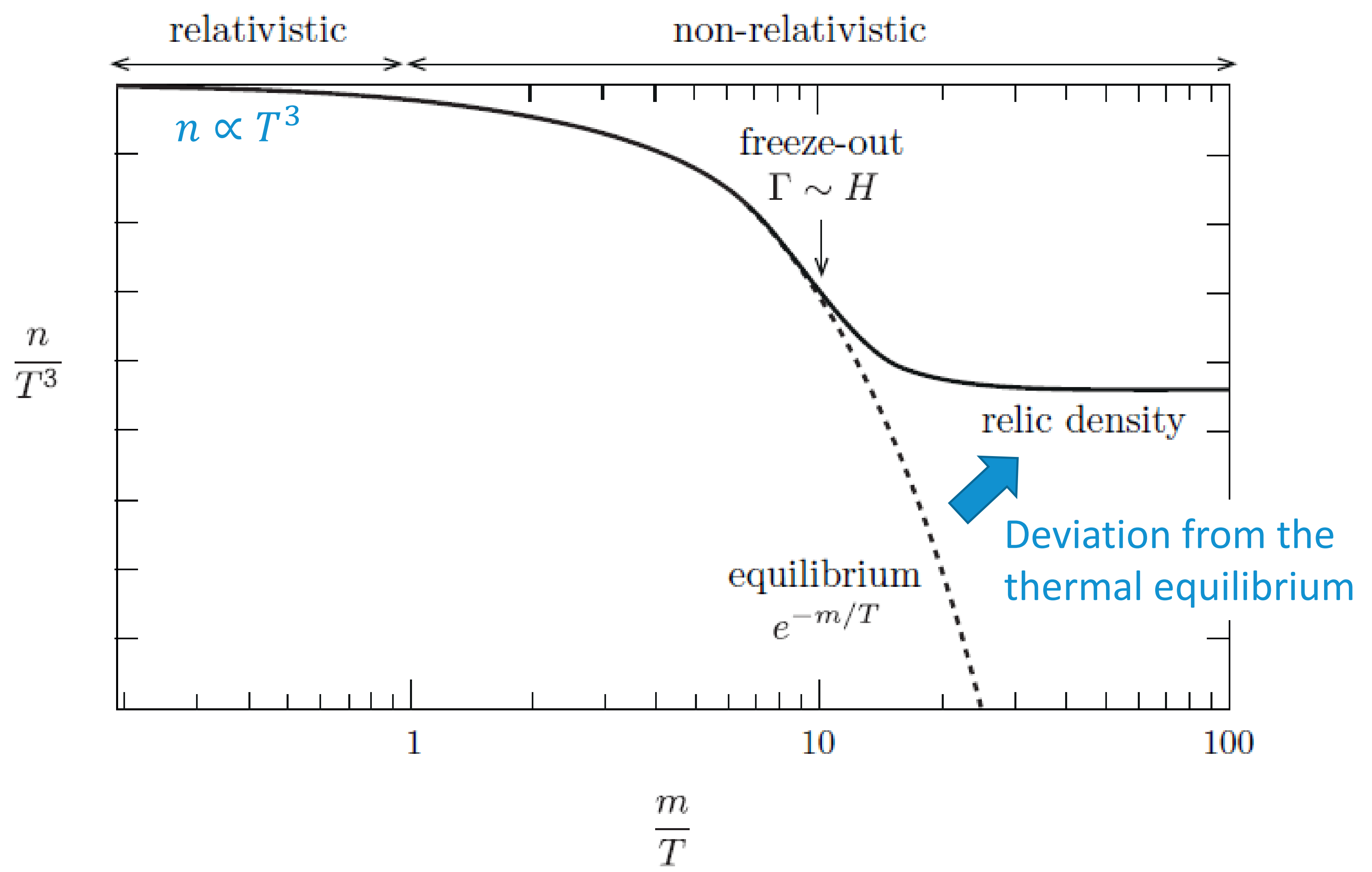
Number of total relativistic DOF *vs.* temperature

All the particles of the Standard Model are relativistic



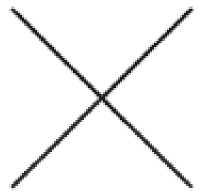
Only photons and (maybe) neutrinos are relativistic

Decoupling and freeze-out of massive particles



Decoupling of weak scale interactions

- Below $T \lesssim 100\ \mathrm{GeV}$ (the scale of electroweak symmetry breaking), the $W^{\pm}$ and Z bosons receive masses ($M_W = 80\ \mathrm{GeV}$ and $M_Z = 90\ \mathrm{GeV}$)
- The cross-section associated with the weak force

$\sigma \sim \Big|$  $\Big|^2 \sim G_F^2 T^2 \sim \dfrac{\alpha^2 T^2}{M_W^4}$

$G_F \sim \alpha / M_W^2 \sim 10^{-5}\ \mathrm{GeV}^{-2}$ (Fermi's constant)

$$\Gamma \propto G_F^2 T^5$$

$$H \propto \frac{T^2}{M_{\mathrm{Pl}}}$$

$$\frac{\Gamma}{H} \propto \frac{\alpha^2 M_{\mathrm{Pl}} T^3}{M_W^4} \propto \left(\frac{T}{1\ \mathrm{MeV}}\right)^3$$

- Particles interacting with the primordial plasma only through the weak interaction should decouple at $\sim 1\ \mathrm{MeV}$

History of the Universe

Models to explain overabundance of matter over anti-matter w/o assuming primordial matter-antimatter asymmetry

Particles receive masses through the Higgs mechanism

Strong interaction between quarks and gluons leads to the formation of baryons and mesons

Is expected to decouple relatively early

Interact with the rest of the primordial plasma through the weak interaction

Energies of the e^- and e^+ are transferred to the photons, but not the neutrinos

Formation of light chemical elements

Formation of neutral H atoms

Small number of remaining free *e* leads to the decoupling of photons and CMB

UV radiation from the first stars reionized the Universe

Event	time t	redshift z	temperature T
Singularity	**0**	∞	∞
Quantum gravity	$\sim 10^{-43}$ s	–	$\sim 10^{18}$ GeV
Inflation	$\gtrsim 10^{-34}$ s	–	–
Baryogenesis	$\lesssim 20$ ps	$> 10^{15}$	> 100 GeV
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1100	0.26 eV
Reionization	100–400 Myr	10–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	**13.8 Gyr**	**0**	**0.24 meV**

Equilibrium Thermodynamics

- Equilibrium distribution function for a particle to have a momentum $\mathbf{p}$:

$$f(\mathbf{p},T)=\frac{1}{\exp\left(\frac{E(p)-\mu}{T}\right)\pm 1}$$

"+" : Fermi–Dirac distribution
"–" : Bose–Einstein distribution
μ – chemical potential describing the response to a change in particle number

- Particle has g internal degrees of freedom
- Density of states in 3D is $\frac{g}{(2\pi\hbar)^3}\equiv\frac{g}{(2\pi)^3}$:

Number density of particles

$$n=g\cdot\sum_{\mathbf{p}} f(\mathbf{p},T)\equiv\frac{g}{(2\pi)^3}\int \mathrm{d}^3p\, f(\mathbf{p},T)$$

Energy density

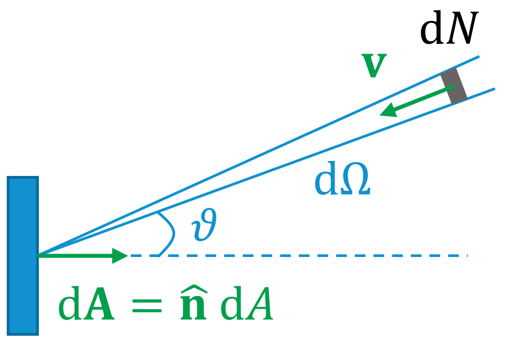
$$\rho=\frac{g}{(2\pi)^3}\int \mathrm{d}^3p\, f(\mathbf{p},T)\cdot E(p)$$

$$\boxed{E(p)=\sqrt{m^2+p^2}}$$

Pressure

- Total number of particles with energy E within the solid angle $\mathrm{d}\Omega$, that will hit this area $\mathrm{d}A$ within the time interval between t and $t + \mathrm{d}t$:

$$\mathrm{d}N_A = \frac{g}{(2\pi)^3} f(E) \cdot \frac{\mathrm{d}\mathbf{A} \cdot \mathbf{v}\mathrm{d}t\, \mathrm{d}\Omega}{4\pi}$$



- Elastic hits transfer the momentum $2|\mathbf{p} \cdot \hat{\mathbf{n}}|$
- The contribution of the particles with velocity $|\mathbf{v}| = |\mathbf{p}|/E$ to the pressure is

$$P(E) = \iint \frac{2|\mathbf{p} \cdot \hat{\mathbf{n}}|}{\mathrm{d}A\mathrm{d}t} \mathrm{d}N_A = \frac{g}{(2\pi)^3} f(E) \cdot \frac{p^2}{\cancel{2\pi} E} \underbrace{\int_0^{\frac{\pi}{2}} \cos^2\vartheta \sin\vartheta\, \mathrm{d}\vartheta \cancel{\int_0^{2\pi} \mathrm{d}\varphi}}_{= 1/3}$$

$$\Rightarrow \quad P = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p\, f(\mathbf{p}, T) \cdot \frac{p^2}{3E}$$

Chemical potential

- Entropy $\mathrm{d}S = \frac{\mathrm{d}U + P\mathrm{d}V - \mu \mathrm{d}N}{T}$ ➡ $\boxed{\mu = -T\left(\frac{\partial S}{\partial N}\right)_{U,V}}$

  or, alternatively $\boxed{\mu = \left(\frac{\partial U}{\partial N}\right)_{S,V}}$

- For a general reaction $A + B \leftrightarrow C + D$ particles flow to the side of the reaction, where the total chemical potential ($\mu_A + \mu_B$ or $\mu_C + \mu_D$) is lower
- In equilibrium, when forward and backward reaction total rates are equal, μ remains constant: $\mu_A + \mu_B = \mu_C + \mu_D$
- Number of photons is not conserved ➡ $\mu_\gamma = 0$
- In the annihilation reaction: $X + \bar{X} \leftrightarrow \gamma + \gamma$

  ↳ $\mu_X = -\mu_{\bar{X}}$

Particle and energy densities

"+" : Fermi–Dirac distribution
"–" : Bose–Einstein distribution

[Ex. to show that for electrons]

- At early times, the chemical potentials of all particles are very small, $\mu \approx 0$
- Particle density

$\boxed{x = \frac{m}{T}, \xi = \frac{p}{T}}$

$$n = \frac{g}{(2\pi)^3} \int \mathrm{d}^3p\, f(\mathbf{p}, T) = \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi p^2\, \mathrm{d}p}{\exp\left(\frac{\sqrt{p^2 + m^2}}{T}\right) \pm 1} = \frac{g}{2\pi^2} T^3 \cdot I_\pm(x)$$

- Energy density

$$\rho = \frac{g}{(2\pi)^3} \int \mathrm{d}^3p\, E f(\mathbf{p}, T) = \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi p^2 \sqrt{p^2 + m^2}\, \mathrm{d}p}{\exp\left(\frac{\sqrt{p^2 + m^2}}{T}\right) \pm 1} = \frac{g}{2\pi^2} T^4 \cdot J_\pm(x)$$

$$I_\pm(x) = \int_0^\infty \frac{\xi^2\, \mathrm{d}\xi}{\exp\left(\sqrt{\xi^2 + x^2}\right) \pm 1}$$

$$J_\pm(x) = \int_0^\infty \frac{\xi^2 \sqrt{\xi^2 + x^2}\mathrm{d}\xi}{\exp\left(\sqrt{\xi^2 + x^2}\right) \pm 1}$$

Particle and energy densities

"+" : Fermi–Dirac distribution
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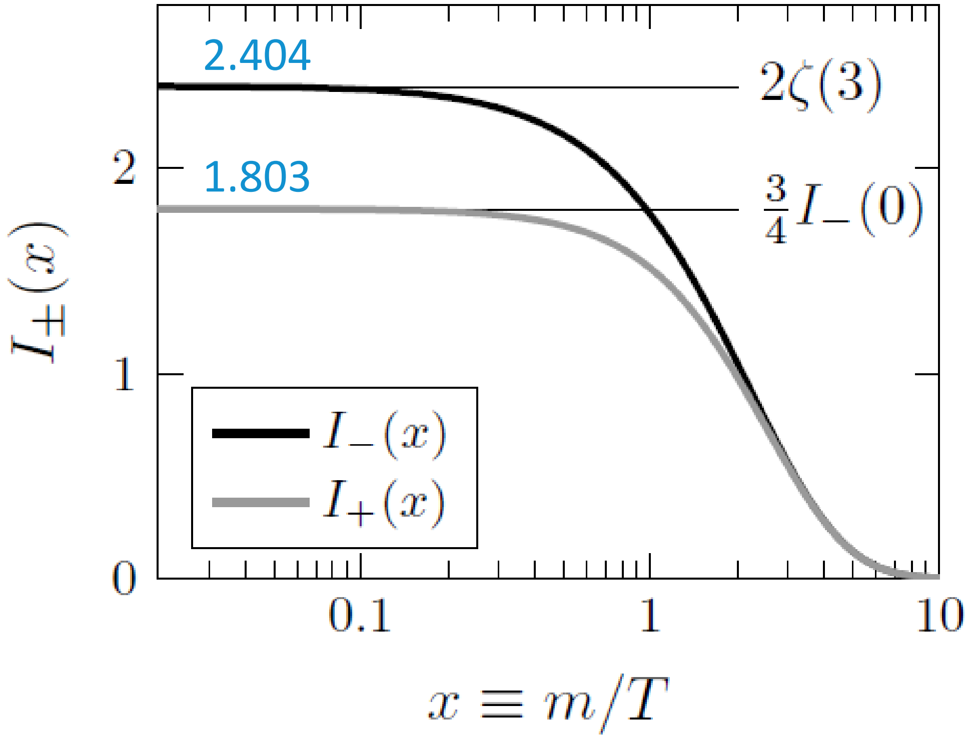
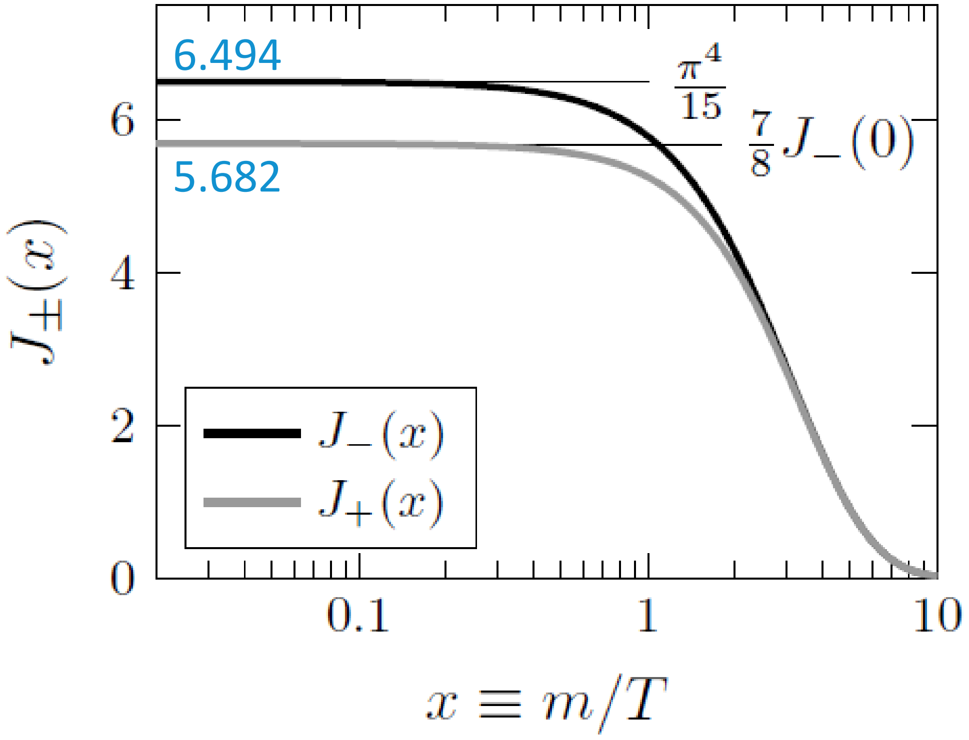
$$I_\pm(x) = \int_0^\infty \frac{\xi^2\,\mathrm{d}\xi}{\exp\left(\sqrt{\xi^2+x^2}\right)\pm 1}$$

$$J_\pm(x) = \int_0^\infty \frac{\xi^2\sqrt{\xi^2+x^2}\,\mathrm{d}\xi}{\exp\left(\sqrt{\xi^2+x^2}\right)\pm 1}$$

$$x = \frac{m}{T}, \xi = \frac{p}{T}$$





Riemann zeta function $\zeta(x) = 1 + 2^{-x} + 3^{-x} + 4^{-x} + \cdots$

Relativistic limit

- $m \ll T,\ x \to 0,\ I_\pm(0) = \int_0^\infty \frac{\xi^2\,\mathrm{d}\xi}{\exp(\xi)\pm 1} = \begin{cases} \frac{3}{2}\zeta(3), & "+", \\ 2\zeta(3), & "-", \end{cases} \qquad J_\pm(0) = \begin{cases} \frac{21}{4}\zeta(4), & "+" \\ 6\zeta(4), & "-" \end{cases}$

$$\boxed{n = \frac{\zeta(3)}{\pi^2} g T^3 \cdot \begin{cases} 1, & \text{bosons} \\ \frac{3}{4}, & \text{fermions} \end{cases}} \qquad \boxed{\rho = \frac{\pi^2}{30} g T^4 \cdot \begin{cases} 1, & \text{bosons} \\ \frac{7}{8}, & \text{fermions} \end{cases}} \qquad \zeta(4) = \frac{\pi^4}{90}$$

- Relic photons: $g = 2, T_0 = 2.73\ \mathrm{K}$
  - Photon density $n_{\gamma,0} = \frac{\zeta(3)}{\pi^2} 2T_0^3 \equiv \frac{\zeta(3)}{\pi^2} 2\left(\frac{k_\mathrm{B} T_0}{\hbar c}\right)^3 \approx 410$ photons/cm³
  - Mass density $\rho_{\gamma,0} = \frac{\pi^2}{30} 2T_0^4 \equiv \frac{\pi^2}{30} 2\frac{(k_\mathrm{B} T_0)^4}{\hbar^3 c^5} \approx 4.64 \cdot 10^{-34}$ g/cm³
  - Dimensionless density $\Omega_{\gamma,0} h^2 = \frac{\rho_{\gamma,0}}{\rho_{\mathrm{crit},0}} = 2.5 \cdot 10^{-5}$
  - Pressure $P_\gamma = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p\, f(\mathbf{p}, T) \cdot \frac{p^2}{3E} = \frac{g}{2\pi^2} \int_0^\infty \frac{p^2}{3p} \frac{p^2\,\mathrm{d}p}{\exp\left(\frac{p}{T}\right) - 1} = \frac{1}{3}\rho_\gamma$

Non-relativistic limit: $m \gg T,\ x \gg 1$

$$x = \frac{m}{T}, \xi = \frac{p}{T}$$

$$I_\pm(x) = \int_0^\infty \frac{\xi^2\,\mathrm{d}\xi}{\exp\left(\sqrt{\xi^2 + x^2}\right) \cancel{\pm 1}} \approx \int_0^\infty \frac{\xi^2\,\mathrm{d}\xi}{\exp\left(x + \frac{1}{2}\frac{\xi^2}{x}\right)} = \mathrm{e}^{-x} \underbrace{\int_0^\infty \xi^2 \exp\left(-\frac{1}{2}\frac{\xi^2}{x}\right) \mathrm{d}\xi}_{= \sqrt{\frac{\pi}{2}} x^{3/2}}$$

$$= \sqrt{\frac{\pi}{2}} x^{3/2} \mathrm{e}^{-x}$$

- Particle density $n = \frac{g}{2\pi^2} T^3 \cdot I_\pm(x) = g\left(\frac{mT}{2\pi}\right)^{3/2} \boxed{\mathrm{e}^{-m/T}}$

Exponential drop of particle number – annihilation of particles and anti-particles

- Energy density $\rho = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p\, \underbrace{\boxed{E}}_{E \approx m} f(\mathbf{p}, T) \approx mn$

- Pressure $P = \frac{g}{(2\pi)^3} \int \mathrm{d}^3 p\, f(\mathbf{p}, T) \cdot \underbrace{\boxed{\frac{p^2}{3E}}}_{\approx \frac{1}{3}\frac{p^2}{m}} \approx \frac{gT^3}{2\pi^2} \frac{T}{3x} \underbrace{\int_0^\infty \frac{\xi^4\,\mathrm{d}\xi}{\exp\left(x + \frac{1}{2}\frac{\xi^2}{x}\right)}}_{= 3\sqrt{\frac{\pi}{2}} x^{5/2} \mathrm{e}^{-x}}$

$$= gT\left(\frac{mT}{2\pi}\right)^{3/2} \mathrm{e}^{-m/T} = nT$$ (pressureless dust, $P = nT \ll \rho = nm$)

Effective number of relativistic species

$$\rho^{(\mathrm{r})} = \frac{\pi^2}{30} g T^4 \cdot \begin{cases} 1, & \text{bosons} \\ \frac{7}{8}, & \text{fermions} \end{cases}$$

- The total energy density of all the particles is

$$\rho = \sum_i \frac{g_i}{2\pi^2} T_i^4 J_\pm(x_i)$$

- It is common to write ρ in terms of the "Temperature of the Universe", which is typically chosen as the photon temperature:

$$\rho = \frac{\pi^2}{30} g_*(T) T^4, \qquad g_*(T) = \sum_i g_i \left(\frac{T_i}{T}\right)^4 \frac{J_\pm(x_i)}{J_-(0)}$$

  - Due to exponential drop-out it is sufficient to include only relativistic species
  - For $T_i \gg m_i$, we have $J_\pm(x_i \ll 1) \approx \mathrm{const}$

Relativistic particles in thermal equilibrium with photons, $T_i = T$:

$$g_*^{(\mathrm{th})}(T) = \sum_{i \in \mathrm{bos}} g_i + \frac{7}{8} \sum_{i \in \mathrm{fer}} g_i$$

Relativistic particles decoupled from the thermal bath (neutrinos after e^+e^- annihilation):

$$g_*^{(\mathrm{dec})}(T) = \sum_{i \in \mathrm{bos}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i \in \mathrm{fer}} g_i \left(\frac{T_i}{T}\right)^4$$

Degrees of freedom

$$g_*^{(\mathrm{th})}(T) = \sum_{i \in \mathrm{bos}} g_i + \frac{7}{8} \sum_{i \in \mathrm{fer}} g_i$$

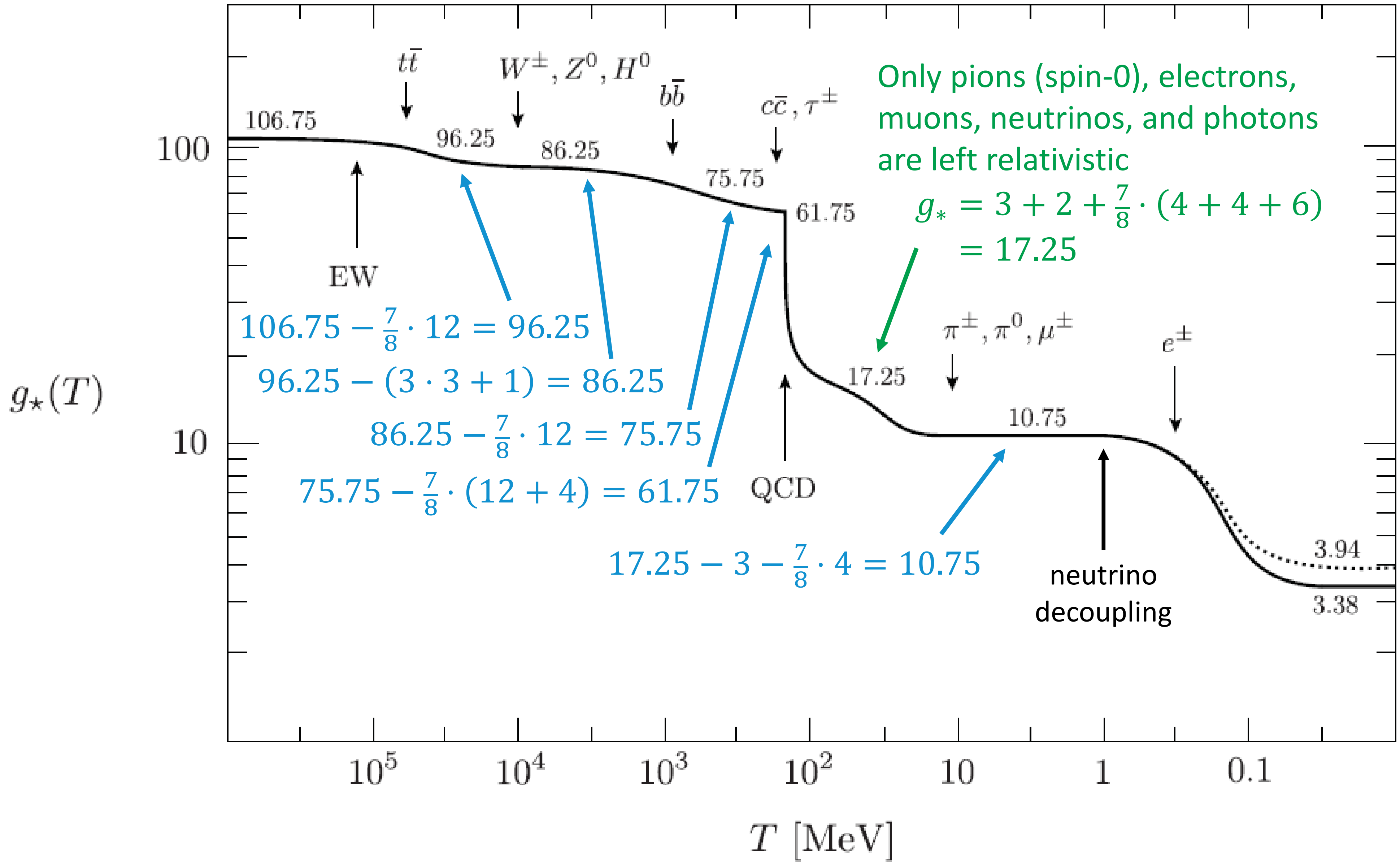
		Mass	Spin	g
quarks	$t, \bar{t}$	$173\ \mathrm{GeV}$	$\frac{1}{2}$	$2 \cdot 2 \cdot 3 = 12$
	$b, \bar{b}$	$4\ \mathrm{GeV}$		Particle and anti-particle
	$c, \bar{c}$	$1\ \mathrm{GeV}$		2 spin states
	$s, \bar{s}$	$100\ \mathrm{MeV}$		3 colors
	$d, \bar{s}$	$5\ \mathrm{MeV}$		
	$u, \bar{u}$	$2\ \mathrm{MeV}$		
leptons	$\tau^{\pm}$	$1777\ \mathrm{MeV}$	$\frac{1}{2}$	$2 \cdot 2 = 4$
	$\mu^{\pm}$	$106\ \mathrm{MeV}$		Particle and anti-particle
	$e^{\pm}$	$511\ \mathrm{keV}$		2 spin states
neutrinos	$\nu_\tau, \bar{\nu}_\tau$	$< 0.6\ \mathrm{eV}$	$\frac{1}{2}$	$2 \cdot 1 = 2$
	$\nu_\mu, \bar{\nu}_\mu$	$< 0.6\ \mathrm{eV}$		*Either*
	$\nu_e, \bar{\nu}_e$	$< 0.6\ \mathrm{eV}$		Particle = anti-particle
				or
				a single spin state exists

		Mass	Spin	g
gauge bosons	W^+	$80\ \mathrm{GeV}$	1	3
	W^-	$80\ \mathrm{GeV}$		3 spin states
	Z^0	$91\ \mathrm{GeV}$		
	γ	0		2
				2 polarization states
gluons	g_i	0	1	$8 \cdot 2 = 16$
				2 polarization states
Higgs boson	H^0	$125\ \mathrm{GeV}$	0	1

At $T > 100$ GeV, all particles are relativistic, hence

$$g_* = 3 \cdot 3 + 2 + 16 + 1 + \frac{7}{8} \cdot (6 \cdot 12 + 3 \cdot 4 + 3 \cdot 2) = 106.75$$

Number of total relativistic DOF *vs.* temperature



Conservation of entropy

$$\rho = \frac{\pi^2}{30} g_*(T) T^4$$

- Total entropy of the Universe only increases or stays constant
- $T\,\mathrm{d}S = \mathrm{d}U + P\,\mathrm{d}V$ ➡ $\displaystyle \frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}t}(\rho V) + \frac{P}{T}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{V}{T}\underbrace{\frac{\mathrm{d}\rho}{\mathrm{d}t}}_{= -3H(\rho+P)} + \frac{\rho + P}{T}\underbrace{\frac{\mathrm{d}V}{\mathrm{d}t}}_{= \frac{\mathrm{d}(a^3)}{\mathrm{d}t} = 3a^2\dot{a} = 3HV} = 0$

  from the continuity equation

- Entropy density $s = S/V$

$$Ts\,\mathrm{d}V + TV\,\mathrm{d}s = \rho\,\mathrm{d}V + V\,\mathrm{d}\rho + P\,\mathrm{d}V$$

$$\rightarrow (Ts - \rho - P)\,\mathrm{d}V + V(T\mathrm{d}s - \mathrm{d}\rho) = 0$$

- s and ρ do not depend on volume, thus $Ts - \rho - P = 0$ ➡ $\boxed{s = \dfrac{\rho + P}{T}}$
- Total entropy: $\displaystyle s = \sum_i \frac{\rho_i + P_i}{T_i} = \sum_i \frac{\rho_i + \frac{1}{3}\rho_i}{T_i} \equiv \frac{2\pi^2}{45} g_{*S}(T) T^3$

Effective number of degrees of freedom of entropy

Particles in thermal equilibrium with photons:

$$g_{*S}^{(\mathrm{th})}(T) = \sum_{i\in\mathrm{bos}} g_i + \frac{7}{8}\sum_{i\in\mathrm{fer}} g_i = g_*^{(\mathrm{th})}(T),$$

Particles decoupled from the thermal bath:

$$g_{*S}^{(\mathrm{dec})}(T) = \sum_{i\in\mathrm{bos}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i\in\mathrm{fer}} g_i \left(\frac{T_i}{T}\right)^3 \neq g_*^{(\mathrm{th})}(T)$$

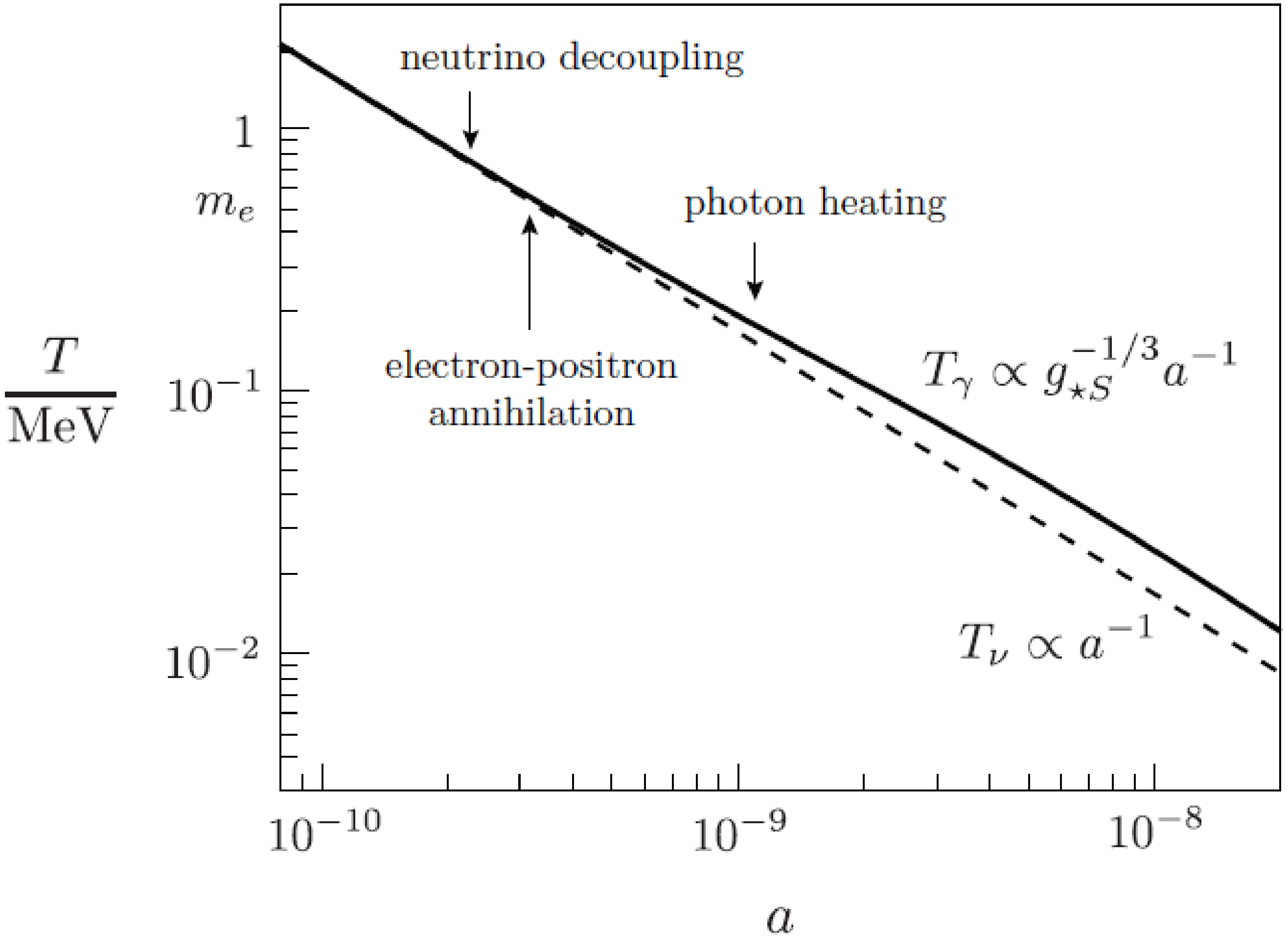
Conservation of entropy

- ❖ $S = sV = \text{const}$ ➡ $s \propto a^{-3}$
  - ➢ Number of particles in a comoving volume $N_i = \frac{n_i}{s}$
- ❖ $s \propto g_{*S}(T) \cdot T^3$ ➡ $g_{*S}(T) \cdot T^3 \cdot a^3 = \text{const}$, or $T \propto g_{*S}^{-1/3} a^{-1}$
  - ➢ Whenever a particle species becomes non-relativistic, its entropy is transferred to the remaining relativistic species, causing the temperature to decrease slightly slower than $\propto a^{-1}$

  - ➢ Hubble constant for the radiation-dominated universe ($\rho = \frac{\pi^2}{30} g_*(T) T^4$):

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_{\rm pl}^2} = \frac{\pi^2}{90} g_*(T) \frac{T^4}{M_{\rm pl}^2}$$

Electron–positron annihilation, $e^+ + e^- \leftrightarrow \gamma + \gamma$



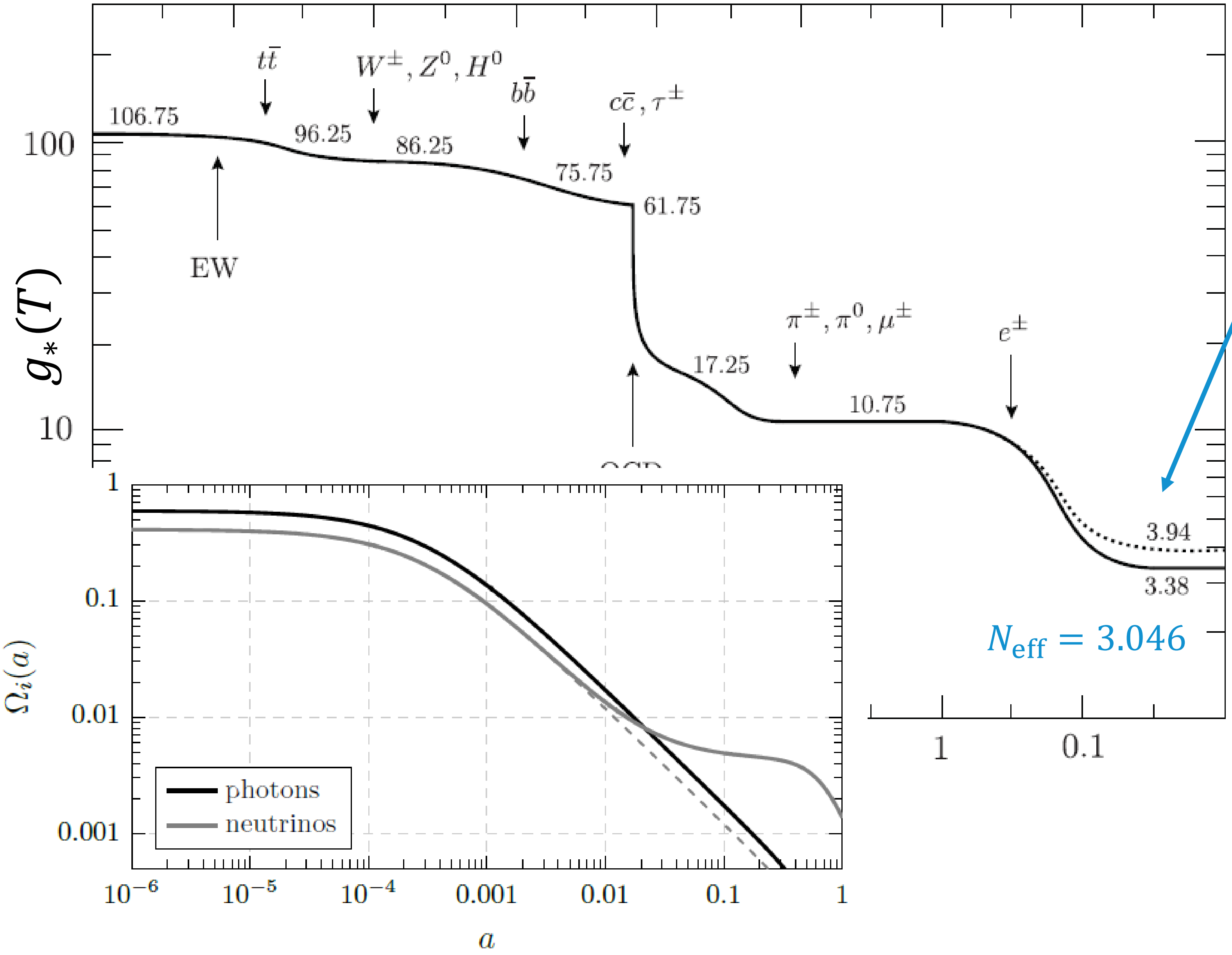
- Energy density and entropy of electrons and positrons is transferred to the photons
  - Photons are heated – their temperature decreases slower than that of the already de-coupled neutrinos
  - $g_{*S}^{(\mathrm{th})} = \begin{cases} 2 + \frac{7}{8} \cdot 4 = \frac{11}{2}, & T \gtrsim m_\mathrm{e} \\ 2, & T < m_\mathrm{e} \end{cases}$
  - After e^+e^- annihilation,

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

  - Current temperature of the cosmic neutrino background $T_{\nu,0} = 1.95\ \mathrm{K}\ \ (T_0 = 2.73\ \mathrm{K})$

Neutrino density

$$g_{*[S]}^{(\mathrm{dec})}(T) = \sum_{i\in\mathrm{bos}} g_i \left(\frac{T_i}{T}\right)^{4\,[3]} + \frac{7}{8}\sum_{i\in\mathrm{fer}} g_i \left(\frac{T_i}{T}\right)^{4\,[3]}, \qquad \frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}$$



- $g_* = 2 + \frac{7}{8}\cdot 2N_{\mathrm{eff}}\left(\frac{4}{11}\right)^{4/3} = 3.36$ $\quad (N_{\mathrm{eff}} = 3)$
- $g_{*S} = 2 + \frac{7}{8}\cdot 2N_{\mathrm{eff}}\,\frac{4}{11} = 3.94$
- Number density:
  $$n_\nu = \frac{3}{4}N_{\mathrm{eff}}\cdot\frac{4}{11}n_\gamma$$
- Energy density:

$m_\nu = 0$: $\quad \rho_\nu = \frac{7}{8}N_{\mathrm{eff}}\cdot\left(\frac{4}{11}\right)^{4/3} n_\gamma$

$$\Omega_\nu h^2 \approx 1.7\cdot 10^{-5}$$

$m_\nu \neq 0$: $\quad \rho_\nu = \sum_i m_{\nu,i}\, n_{\nu,i}$

$$\Omega_\nu h^2 \approx \frac{\sum_i m_{\nu,i}}{94\ \mathrm{eV}} < 0.02$$

Beyond Equilibrium

- Boltzmann equation
  - In the absence of interactions the number of particles is conserved:

$$\frac{1}{a^3}\frac{\mathrm{d}(n_i a^3)}{\mathrm{d}t} = 0$$

  - To account for the interactions, we include the collision term:

$$\frac{1}{a^3}\frac{\mathrm{d}(n_i a^3)}{\mathrm{d}t} = C_i[\{n_i\}]$$

Boltzmann equation

- Example: $A + B \leftrightarrow C + D$

$\alpha = \langle \sigma v \rangle$ – thermally averaged cross-section

$$\frac{1}{a^3}\frac{\mathrm{d}(n_A a^3)}{\mathrm{d}t} = -\boxed{\alpha} n_A n_B + \beta n_C n_D$$

  - In (chemical) equilibrium: $[\alpha n_A n_B]_{\mathrm{eq}} = [\beta n_C n_D]_{\mathrm{eq}}$ ➡ $\beta = \alpha \cdot \left(\frac{n_A n_B}{n_C n_D}\right)_{\mathrm{eq}}$

$$\frac{1}{a^3}\frac{\mathrm{d}(n_A a^3)}{\mathrm{d}t} = -\langle \sigma v \rangle \left[n_A n_B - \left(\frac{n_A n_B}{n_C n_D}\right)_{\mathrm{eq}} n_C n_D \right]$$

History of the Universe

Event	time t	redshift z	temperature T
Singularity	**0**	∞	∞
Quantum gravity	$\sim 10^{-43}$ s	–	$\sim 10^{18}$ GeV
Inflation	$\gtrsim 10^{-34}$ s	–	–
Baryogenesis	$\lesssim 20$ ps	$> 10^{15}$	> 100 GeV
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^{9}	1 MeV
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Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	**13.8 Gyr**	**0**	**0.24 meV**

Dark matter relics

- Assumptions:
  - Dark matter consists of weakly interacting massive particles (WIMPs)
  - Heavy dark matter particle X and its antiparticle $\bar{X}$ can annihilate to produce two light (≈massless) particles ℓ and $\bar{\ell}$:
$$X + \bar{X} \leftrightarrow \ell + \bar{\ell}$$
  - ℓ are tightly coupled to the cosmic plasma ➡ $n_\ell \approx n_\ell^{(\mathrm{eq})}$
  - No asymmetry X and $\bar{X}$, i.e. $n_X = n_{\bar{X}}$

Dark matter relics

- Boltzmann equation:

$$\frac{1}{a^3}\frac{\mathrm{d}(n_X a^3)}{\mathrm{d}t} = -\langle\sigma v\rangle\left[n_X^2 - \left(n_X^{(\mathrm{eq})}\right)^2\right]$$

- Let's introduce new quantities:
  - $Y_X = \frac{n_X}{T^3}$ ($\propto N_X = \frac{n_X}{s}$, the number of particles in a comoving volume)
  - $x = \frac{M_x}{T}$ $\Rightarrow$ $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{M_X}{T^2}\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{x}{T}\frac{\mathrm{d}T}{\mathrm{d}t} \overset{T \propto a^{-1}}{=} \frac{x}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = Hx$
- During radiation-dominated era:

$$H = H(T = M_X)\cdot\left[\frac{a(T = M_X)}{a}\right]^2 \overset{T \propto a^{-1}}{=} H(M_X)\cdot\left[\frac{T}{M_X}\right]^2 = \frac{H(M_X)}{x^2}$$

- Riccati equation:

$$\boxed{\frac{\mathrm{d}Y_x}{\mathrm{d}x} = -\frac{\lambda}{x^2}\left[Y_X^2 - \left(Y_X^{(\mathrm{eq})}\right)^2\right]}$$

$$\lambda = \frac{M_X^3}{H(M_X)}\langle\sigma v\rangle$$

Dark matter relics

❖ Boltzmann equatic

➢ Let's introduce new

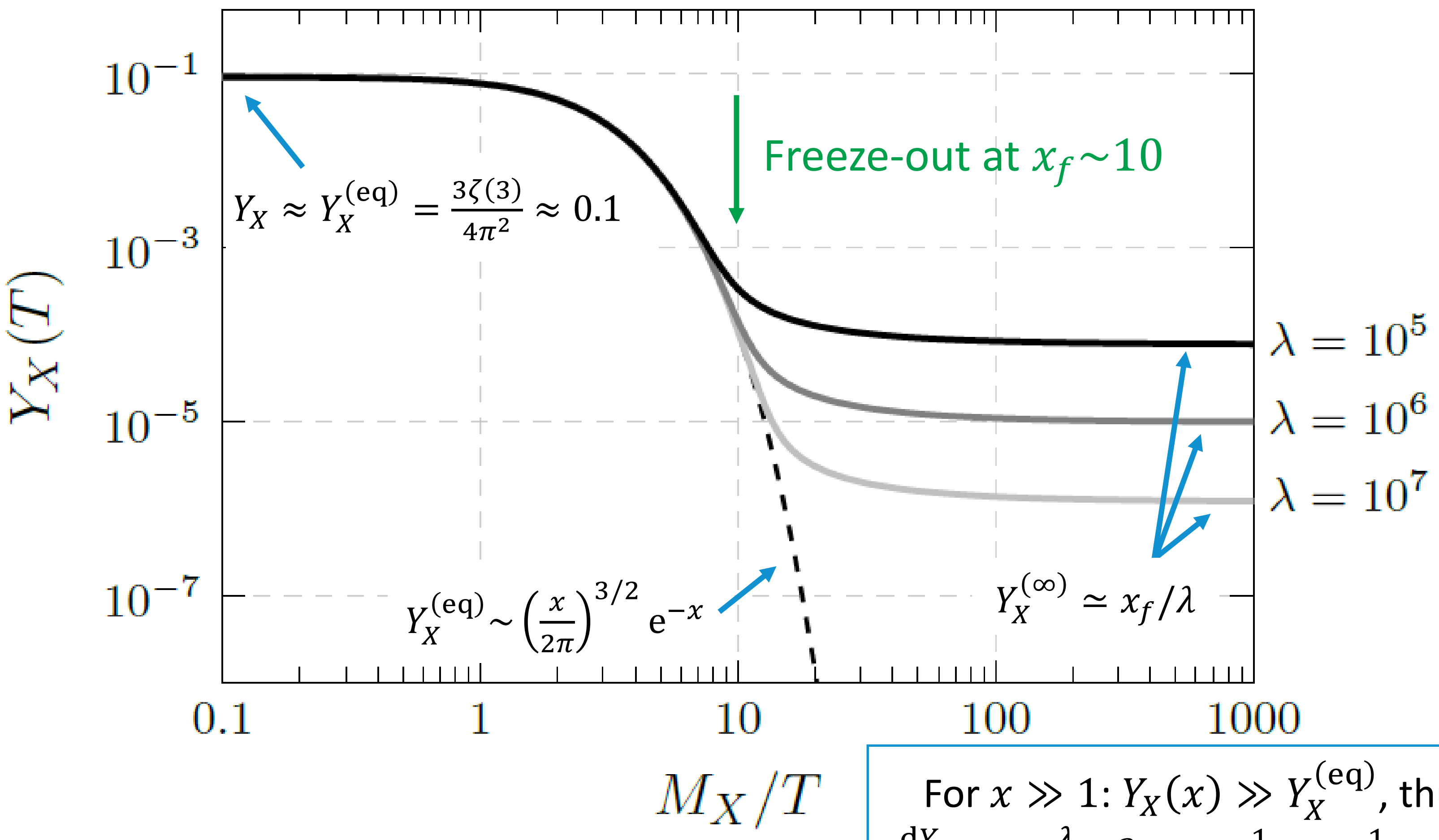
❑ $Y_X = \frac{n_X}{T^3}$ ($\propto N_X = \frac{n}{}$

❑ $x = \frac{M_X}{T}$ ➡ $\frac{\mathrm{d}x}{\mathrm{d}t}$

➢ During radiation-dc

$H = H(T = M_X) \cdot \left[\frac{a(T}{}\right.$

Numerical solution



❖ Riccati equation:

$$\boxed{\frac{\mathrm{d}Y_x}{\mathrm{d}x} = -\frac{\lambda}{x^2}\left[Y_X^2 - \left(Y_X^{(\mathrm{eq})}\right)^2\right]}$$

$$\lambda = \frac{M_X^3}{H(M_X)}\langle\sigma v\rangle \approx \mathrm{const}$$

For $x \gg 1$: $Y_X(x) \gg Y_X^{(\mathrm{eq})}$, thus

$$\frac{\mathrm{d}Y_x}{\mathrm{d}x} \simeq -\frac{\lambda}{x^2} Y_x^2 \Rightarrow \frac{1}{Y_X^{(\infty)}} - \frac{1}{Y_X^{(f)}} = \frac{\lambda}{x_f}$$

⬇

$$Y_X^{(\infty)} \simeq x_f/\lambda$$

Dark matter density

$Y_X = \frac{n_X}{T^3}$, $Y_X^{(\infty)} \simeq \frac{x_f}{\lambda}$, $\lambda = \frac{M_X^3}{H(M_X)}\langle\sigma v\rangle$

- Let's relate the freeze-out abundance of dark matter relics to the dark matter density today

$$\Omega_X = \frac{\rho_{X,0}}{\rho_{\mathrm{crit},0}} = \frac{M_X n_{X,0}}{3M_{\mathrm{pl}}^2 H_0^2} = \frac{M_X}{3M_{\mathrm{pl}}^2 H_0^2} Y_X^{(\infty)} T_0^3 \frac{g_{*S}(T_0)}{g_{*S}(M_X)} = \frac{M_X}{3M_{\mathrm{pl}}^2 H_0^2} \frac{x_f}{\lambda} T_0^3 \frac{g_{*S}(T_0)}{g_{*S}(M_X)}$$

- Number density: $n_{X,0} = n_{X,1}\left(\frac{a_1}{a_0}\right)^3 = Y_X^{(\infty)} T_1^3 \left(\frac{a_1}{a_0}\right)^3 = Y_X^{(\infty)} T_0^3 \left(\frac{a_1 T_1}{a_0 T_0}\right)^3$
- Conservation of entropy: $g_{*S}(aT)^3 = \mathrm{const}$ ⇒ $n_{X,0} = Y_X^{(\infty)} T_0^3 \frac{g_{*S}(T_0)}{g_{*S}(M_X)}$

$$\Omega_X = \frac{H(M_X)}{M_X^2} \frac{T_0^3}{3M_{\mathrm{pl}}^2 H_0^2} \frac{x_f}{\langle\sigma v\rangle} \frac{g_{*S}(T_0)}{g_{*S}(M_X)} \approx 0.1 \frac{x_f}{\sqrt{g_*(M_X)}} \frac{10^{-8}\ \mathrm{GeV}^{-2}}{\langle\sigma v\rangle}$$

≈ 0.27 if $\langle\sigma v\rangle \sim 10^{-8}\ \mathrm{GeV}^{-2}$, a characteristic scale for the weak interaction (***WIMP miracle***)

- $T_0 = 2.73\ \mathrm{K}$, $H_0 = 70\ \mathrm{km\ s^{-1}\ Mpc^{-1}}$, $H^2(M_X) = \frac{\pi^2}{90} g_*(M_X) \frac{M_X^4}{M_{\mathrm{pl}}^2}$, $g_{*S}(T_0) = 3.91$, $g_{*S}(M_X) = g_*(M_X)$

History of the Universe

Event	time t	redshift z	temperature T
Singularity	$\mathbf{0}$	∞	∞
Quantum gravity	$\sim 10^{-43}$ s	–	$\sim 10^{18}$ GeV
Inflation	$\gtrsim 10^{-34}$ s	–	–
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Electron–proton recombination (first H atoms)

- When $T \gtrsim 1$ eV: $\quad e^- + p^+ \leftrightarrow H + \gamma$
- $T \ll m_e \approx 0.5$ MeV, thus equilibrium densities are

$$n_i^{(\mathrm{eq})} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right), \qquad i = \{e, p, H\}$$

  - $\mu_e + \mu_p = \mu_H$

- Let's consider the ratio

$$\left(\frac{n_H}{n_e n_p}\right)_{\mathrm{eq}} = \underbrace{\boxed{\frac{g_H}{g_e g_p}}}_{\approx \frac{4}{2\cdot 2} = 1} \left(\frac{\overbrace{\boxed{m_H}}^{\approx m_p}}{m_e m_p} \frac{2\pi}{T}\right)^{3/2} \exp\left(\frac{\overbrace{\boxed{m_p + m_e - m_H}}^{\approx E_B = 13.6\text{ eV}}}{T}\right)$$

$n_e = n_p$

$$\left(\frac{n_H}{n_e^2}\right)_{\mathrm{eq}} = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left(\frac{E_B}{T}\right)$$

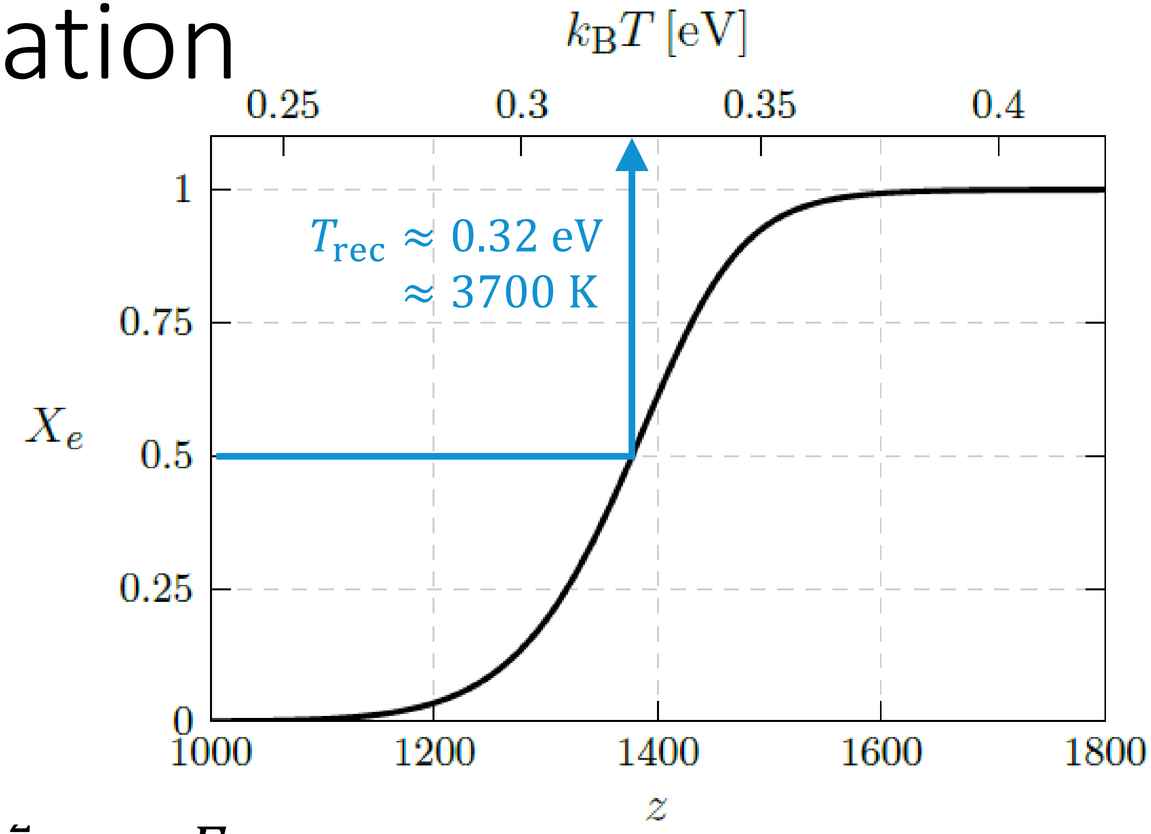
Electron–proton recombination

- $\left(\dfrac{n_H}{n_e^2}\right)_{\text{eq}} = \left(\dfrac{2\pi}{m_e T}\right)^{3/2} \exp\left(\dfrac{E_B}{T}\right)$
- Fraction of free electrons $X_e = n_e/n_b$, here n_b is baryon density
  - $n_b = \eta_b n_\gamma = \eta_b \dfrac{\zeta(3)}{\pi^2} 2T^3$
  - $\eta_b = 5.5 \cdot 10^{-10}$ – baryon-to-photon ratio
  - Ignoring other nuclei, baryon density $n_b \approx n_p + n_H = n_e + n_H$

$$\left(\frac{1 - X_e}{X_e^2}\right)_{\text{eq}} = \frac{2\zeta(3)}{\pi^2}\eta_b \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left(\frac{E_B}{T}\right)$$ Saha equation

$$X_e^{\text{eq}} = \frac{-1 + \sqrt{1+f}}{2f}, \qquad f = \frac{2\zeta(3)}{\pi^2}\eta_b \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left(\frac{E_B}{T}\right)$$

Electron–proton recombination



- $\left(\frac{n_H}{n_e^2}\right)_{eq} = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left(\frac{E_B}{T}\right)$
- Fraction of free electrons $X_e = n_e/n$
  - $n_b = \eta_b n_\gamma = \eta_b \frac{\zeta(3)}{\pi^2} 2T^3$
  - $\eta_b = 5.5 \cdot 10^{-10}$ – baryon-to-photon rat
  - Ignoring other nuclei, baryon density n_b

$$\left(\frac{1 - X_e}{X_e^2}\right)_{eq} = \frac{2\zeta(3)}{\pi^2} \eta_b \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left(\frac{E_B}{T}\right)$$

Saha equation

$$X_e^{eq} = \frac{-1 + \sqrt{1+f}}{2f}, \qquad f = \frac{2\zeta(3)}{\pi^2} \eta_b \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left(\frac{E_B}{T}\right)$$

- The redshift of recombination $1 + z_{rec} = a(t_0)/a(t_{rec}) = T_{rec}/T_0$
- $a(t) = (t/t_0)^{2/3}$ ⇒ $t_{rec} = \frac{t_0}{(1+z_{rec})^{3/2}} = t_0 \left(\frac{T_0}{T_{rec}}\right)^{3/2} \approx 290$ kyr

Photon decoupling

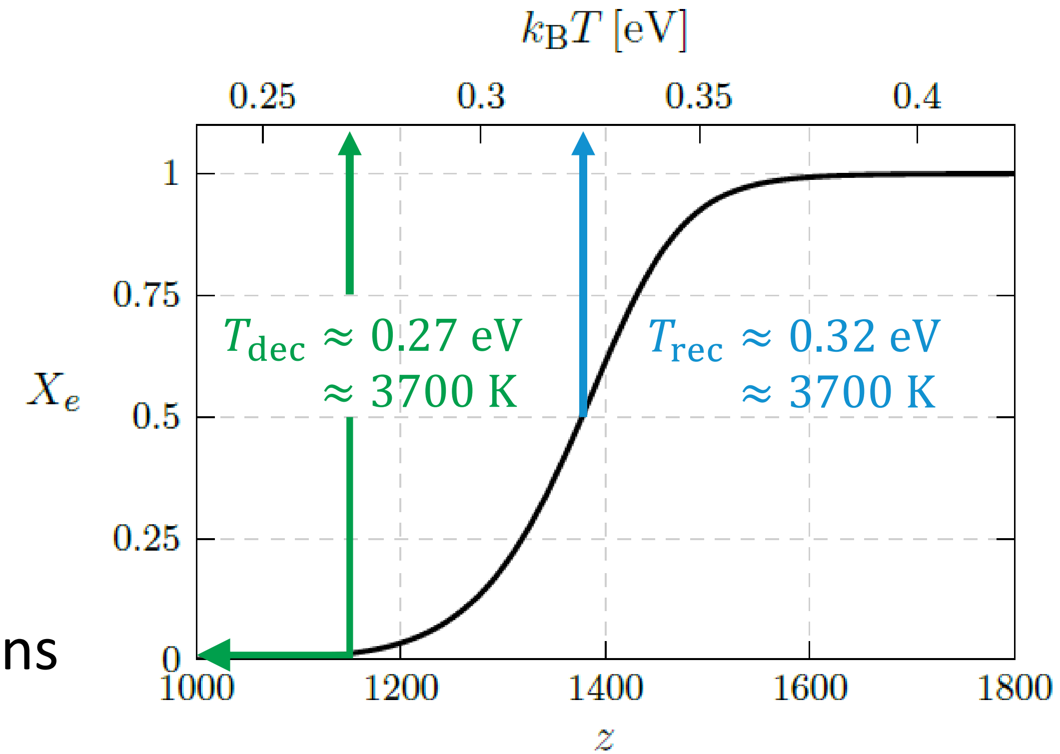
- Interaction with the remaining free electrons: $e^- + \gamma \leftrightarrow e^- + \gamma$
- Interaction rate $\Gamma \approx n_e \sigma_T$ with Thomson cross-section $\sigma_T = 2 \cdot 10^{-3}\ \mathrm{MeV}^{-2}$
- Photons and electrons decouple when $\Gamma(T_{\mathrm{dec}}) \approx H(T_{\mathrm{dec}})$
  - $\Gamma(T_{\mathrm{dec}}) = n_e \sigma_T = n_b X_e(T_{\mathrm{dec}}) \sigma_T = \sigma_T \eta_b \frac{2\zeta(3)}{\pi^2} T_{\mathrm{dec}}^3 X_e(T_{\mathrm{dec}})$
  - $H(T_{\mathrm{dec}}) = H_0 \sqrt{\Omega_m} \left(\frac{T_{\mathrm{dec}}}{T_0}\right)^{3/2}$

$$T_{\mathrm{dec}} \approx 0.27\ \mathrm{eV}$$

$$X_e(T_{\mathrm{dec}}) \approx 0.01$$

$$t_{\mathrm{dec}} \approx 380\ \mathrm{kyr}$$

  - A large degree of neutrality is necessary for the universe to become transparent to photons



Electron freeze-out

- Boltzmann equation:

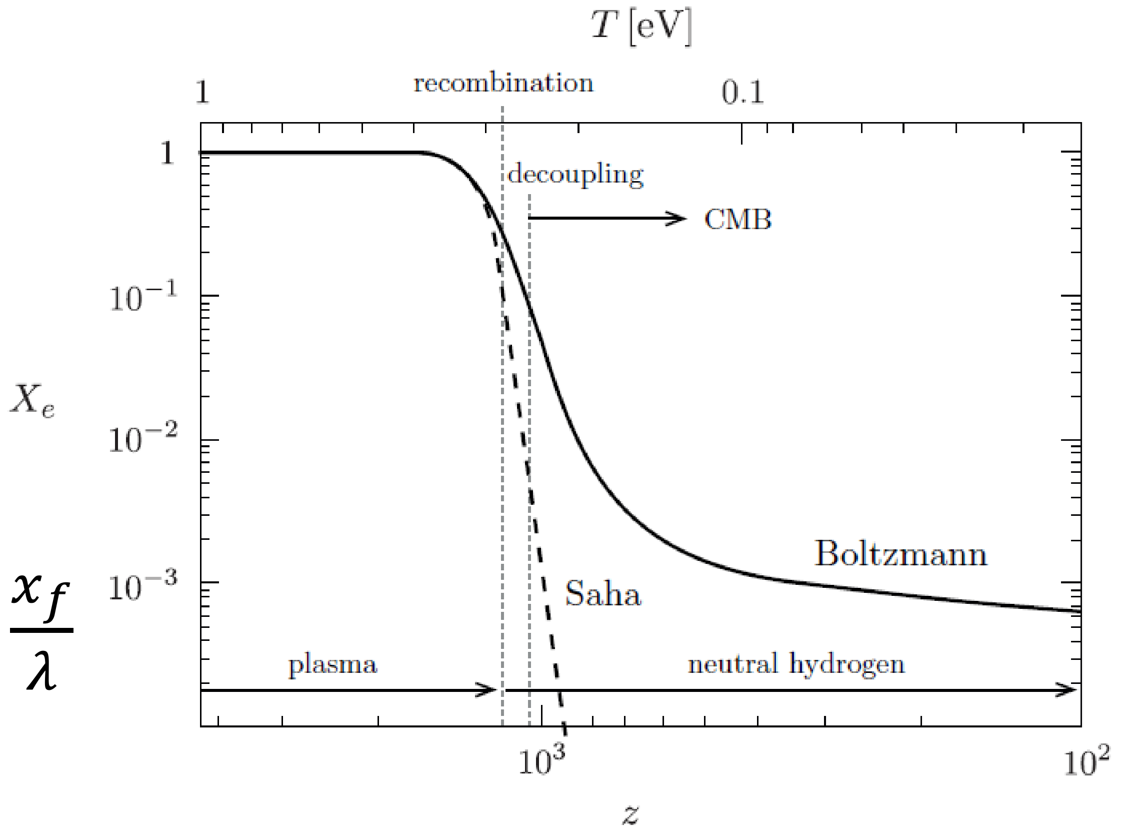
$$\frac{1}{a^3}\frac{\mathrm{d}(n_e a^3)}{\mathrm{d}t} = -\langle\sigma v\rangle\left[n_e^2 - \left(n_e^{(\mathrm{eq})}\right)^2\right]$$

  - $\langle\sigma v\rangle \simeq \sigma_T\sqrt{E_H/T}$
  - $n_e = n_b X_e, \quad n_b a^3 = \mathrm{const}$
  - $x = E_H/T, \quad T \propto g_{*S}^{-1/3} a^{-1}$

$$\frac{\mathrm{d}X_e}{\mathrm{d}x} = -\frac{\lambda}{x^2}\left[X_e^2 - \left(X_e^{(\mathrm{eq})}\right)^2\right]$$

The same eq. as for dark matter freeze-out!

$$\lambda = \left[\frac{n_b}{xH}\langle\sigma v\rangle\right]_{x=1}$$



- Electron freeze-out abundance: $X_e^{(\infty)} \simeq \dfrac{x_f}{\lambda}$

$x_f \approx 54, \; T_f \approx 0.25$ eV

History of the Universe

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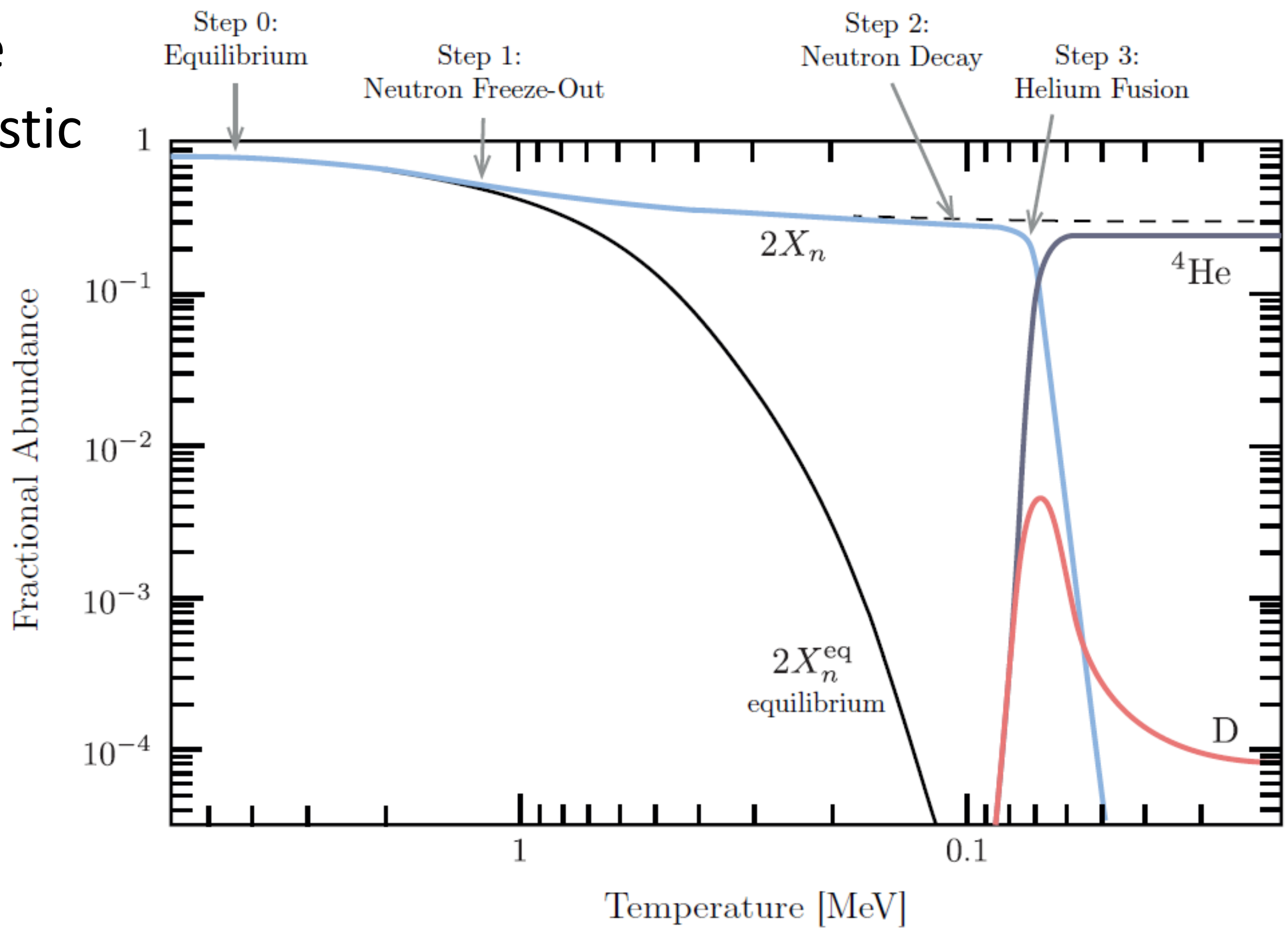
Big Bang Nucleosynthesis

- $T \sim 1$ MeV:
  - Photons, electrons, and positrons are in equilibrium
  - Neutrinos are going to decouple
  - Baryons already are non-relativistic
- Currently, $\dfrac{n_{\mathrm{He}}}{n_{\mathrm{H}}} \sim \dfrac{1}{16}$
  - Why?



Step 0: Equilibrium

- Simplifications:
  - No elements heavier than helium (H, D, T, and ^{3}He, and He)
  - Only neutrons and protons exist at $T > 0.1$ MeV
  - Chemical potentials for e^- and ν_e are negligible
- $n - p$ equilibrium
  - $n + \nu_e \leftrightarrow p^+ + e^-, \quad n + e^+ \leftrightarrow p^+ + \bar{\nu}_e$
  - $\mu_n \approx \mu_n$

$$\boxed{n_i^{(\mathrm{eq})} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right)} \Rightarrow \left(\frac{n_n}{n_p}\right)_{\mathrm{eq}} = \underbrace{\boxed{\left(\frac{m_n}{m_p}\right)^{3/2}}}_{\approx 1} \exp\left(-\frac{\overbrace{\boxed{m_n - m_p}}^{\approx Q = 1.30\ \mathrm{MeV}}}{T}\right) = \exp\left(-\frac{Q}{T}\right)$$

  - For $T < 1$ MeV, the fraction of neutrons gets smaller

Step 0: Equilibrium

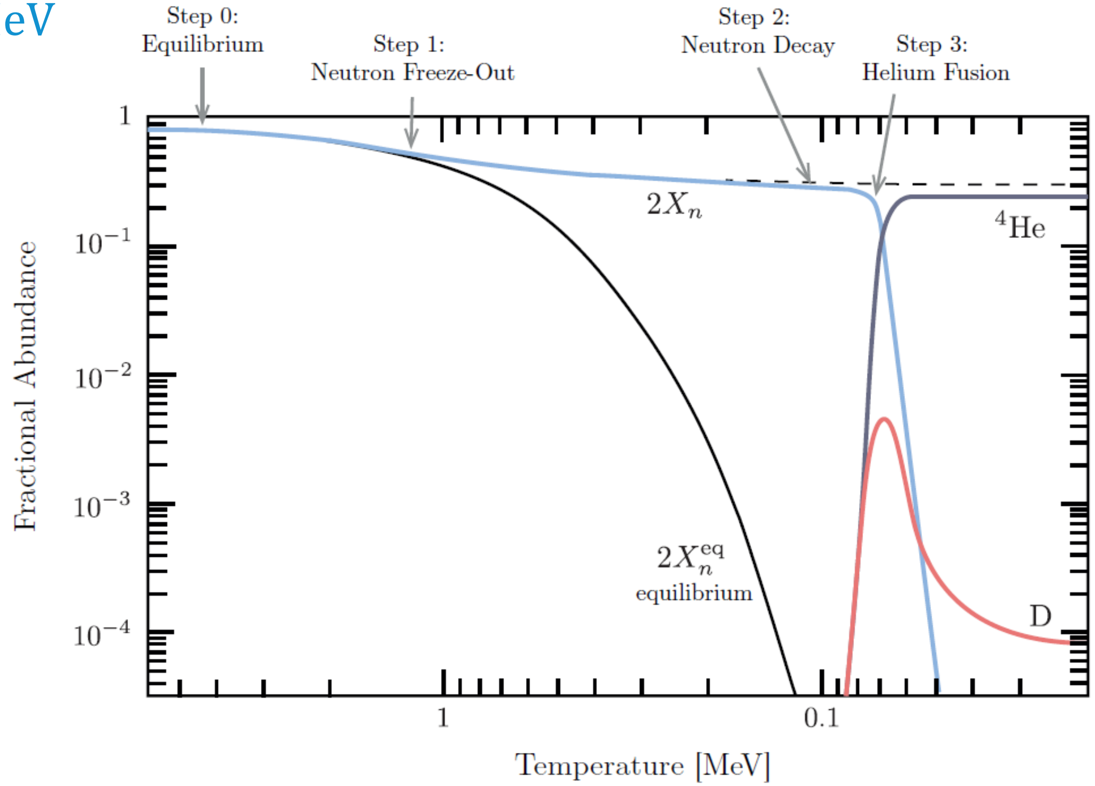
$$n_i^{(\mathrm{eq})} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right)$$

- $n - p$ equilibrium: $\left(\frac{n_n}{n_p}\right)_{\mathrm{eq}} = \exp\left(-\frac{Q}{T}\right)$
- Deuteron:
  - $n + p^+ \leftrightarrow D + \gamma$
  - $\mu_D = \mu_n + \mu_p$

$$\left(\frac{n_D}{n_p n_n}\right)_{\mathrm{eq}} = \frac{3}{4}\left(\underbrace{\boxed{\frac{m_D}{m_n m_p}}}_{\approx 2/m_p}\frac{2\pi}{T}\right)^{3/2} \exp\left(\frac{\overbrace{\boxed{m_n + m_p - m_D}}^{\approx E_D = 2.22\ \mathrm{MeV}}}{T}\right)$$

$$\Downarrow$$

$$\left(\frac{n_D}{n_p}\right)_{\mathrm{eq}} \approx \frac{3}{4} n_n^{\mathrm{eq}} \left(\frac{4\pi}{m_p T}\right)^{3/2} \exp\left(\frac{E_D}{T}\right)$$



Step 1: Neutron Freeze-out

❖ Boltzmann equation ($n + \nu_e \leftrightarrow p^+ + e^-$):

$$\frac{1}{a^3}\frac{\mathrm{d}(n_n a^3)}{\mathrm{d}t} = -\Gamma_n\left[n_n - \left(\frac{n_n}{n_p}\right)_{\mathrm{eq}} n_p\right]$$

$$A + B \leftrightarrow C + D:$$

$$\frac{1}{a^3}\frac{\mathrm{d}(n_A a^3)}{\mathrm{d}t} = -\langle\sigma v\rangle\left[n_A n_B - \left(\frac{n_A n_B}{n_C n_D}\right)_{\mathrm{eq}} n_C n_D\right]$$

- $\Gamma_n = n_\ell\langle\sigma v\rangle$
- $X_n = \frac{n_n}{n_n+n_p}$, $n_n + n_p \sim n_b \propto a^{-3}$, $\left(\frac{n_n}{n_p}\right)_{\mathrm{eq}} = \exp\left(-\frac{Q}{T}\right)$

$$\frac{\mathrm{d}X_n}{\mathrm{d}t} = -\Gamma_n\left[X_n - (1 - X_n)\mathrm{e}^{-Q/T}\right]$$

- $x = Q/T$

$$\frac{\mathrm{d}X_n}{\mathrm{d}x} = -\frac{\Gamma_n}{H_1}x\left[\mathrm{e}^{-x} - X_n(1 + \mathrm{e}^{-x})\right]$$

Numerical solution yields
$X_n^\infty \equiv X_n(x = \infty) = 0.15$

- For $T \sim 1$ MeV, Γ_n^{-1} is comparable with the age of the Universe
- At later times, $T \propto t^{-1/2}$, and $\Gamma_n \propto T^3 \propto t^{-3/2}$, so the neutron-proton conversion time $\Gamma_n^{-1} \propto t^{3/2}$ becomes even longer
- As a result, n_n/n_p ratio approaches a constant

Step 1: Neutron Freeze-out

- Boltzmann equation ($n + \nu_e \leftrightarrow p^+ + e^-$):

$$\frac{1}{a^3}\frac{\mathrm{d}(n_n a^3)}{\mathrm{d}t} = -\Gamma_n\left[n_n - \left(\frac{n_n}{n_p}\right)_{\mathrm{eq}} n_p\right]$$

  - $\Gamma_n = n_\ell \langle \sigma v \rangle$
  - $X_n = \frac{n_n}{n_n+n_p}$, $n_n + n_p \sim n_b \propto a^{-3}$, $\left(\frac{n_n}{n_p}\right)_{\mathrm{eq}}$

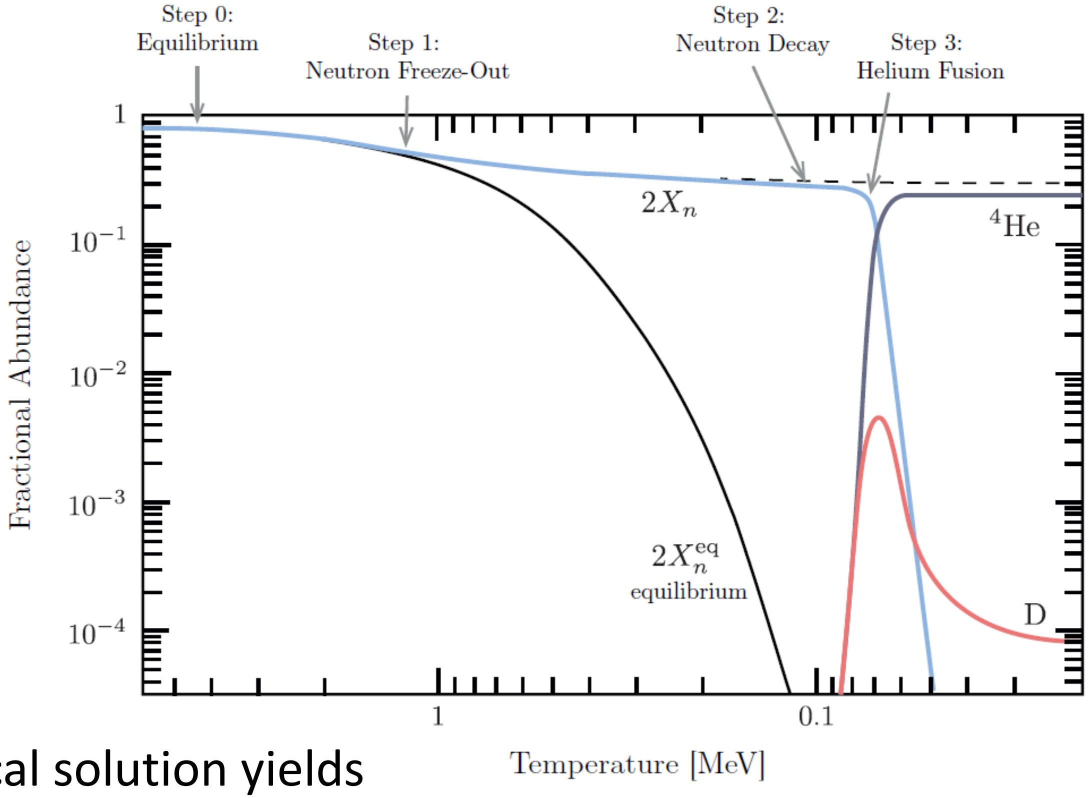
$$\frac{\mathrm{d}X_n}{\mathrm{d}t} = -\Gamma_n\left[X_n - (1 - X_n)\right.$$

  - $x = Q/T$

$$\frac{\mathrm{d}X_n}{\mathrm{d}x} = -\frac{\Gamma_n}{H_1} x\left[\mathrm{e}^{-x} - X_n(1 + \mathrm{e}^{-x})\right]$$

$A + B \leftrightarrow C + D$:

$$\frac{1}{a^3}\frac{\mathrm{d}(n_A a^3)}{\mathrm{d}t} = -\langle \sigma v \rangle\left[n_A n_B - \left(\frac{n_A n_B}{n_C n_D}\right)_{\mathrm{eq}} n_C n_D\right]$$

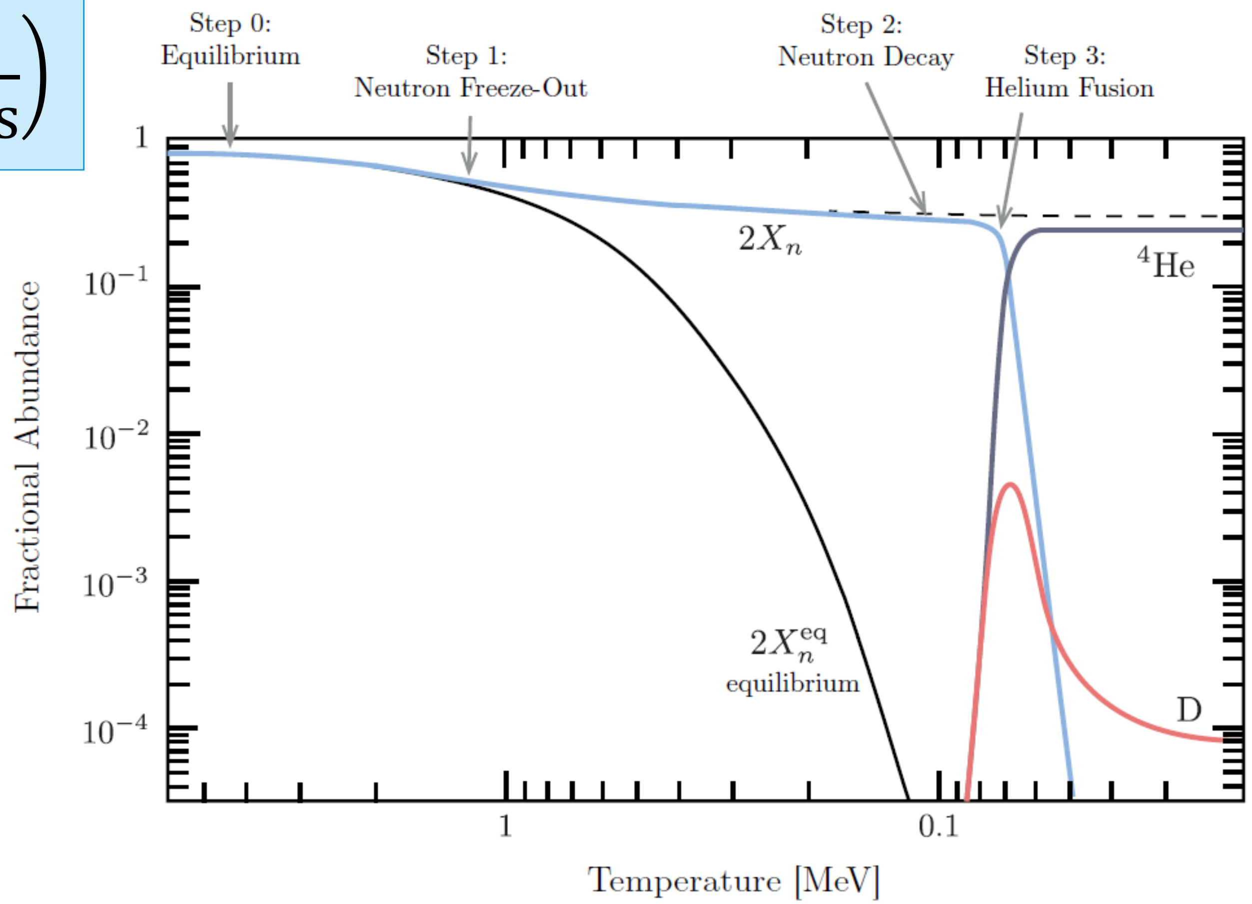


Numerical solution yields
$X_n^\infty \equiv X_n(x = \infty) = 0.15$

Step 2: Neutron Decay

❖ At $T < 0.2$ MeV ($t \gtrsim 100$ s), the finite lifetime of a neutron ($\tau_n = 887$ s) becomes important:

$$X_n(t) = X_n^\infty \exp\left(-\frac{t}{\tau_n}\right) \approx \frac{1}{6}\exp\left(-\frac{t}{887\ \mathrm{s}}\right)$$



Step 3: Helium Fusion

- Deuteron:
  - $n + p^+ \leftrightarrow D + \gamma$
  - n_D follows the n_n and n_p equilibrium abundance
  - $\left(\frac{n_D}{n_p}\right)_{\text{eq}} \approx \frac{3}{4} n_n^{\text{eq}} \left(\frac{4\pi}{m_p T}\right)^{3/2} \exp\left(\frac{E_D}{T}\right) \sim \eta_b \left(\frac{T}{m_p}\right)^{3/2} \exp\left(\frac{E_D}{T}\right) \sim 1$
  - $n_n^{\text{eq}} \sim n_b = \eta_b n_\gamma = \eta_b \cdot \frac{2\zeta(3)}{\pi^2} T^3$, here baryon/photon ratio $\eta_b \sim 10^{-9}$

$T_{\text{nuc}} \approx 0.06\ \text{MeV}$
$t_{\text{nuc}} \approx 330\ \text{s}$

  - $X_n(t_{\text{nuc}}) \sim \frac{1}{6} \exp\left(-\frac{330\ \text{s}}{887\ \text{s}}\right) \sim 0.11$

Step 3: Helium Fusion

❖ Helium:

➢ $D + p^+ \leftrightarrow {}^3\mathrm{He} + \gamma, \quad D + 3\mathrm{He} \leftrightarrow {}^4\mathrm{He} + p^+$

➢ Binding energy of He is larger than that of D (7.1 MeV *vs* 1.1 MeV per nucleon)

➢ Formation of He starts immediately after some D is produced

➢ Virtually all the neutrons are bound in He at $t \sim t_{\mathrm{nuc}}$

➢ Final $n_{\mathrm{He}} = \frac{1}{2} n_n(t_{\mathrm{nuc}})$

$$\frac{n_{\mathrm{He}}}{n_{\mathrm{H}}} = \frac{n_{\mathrm{He}}}{n_p} \approx \frac{\frac{1}{2} X_n(t_{\mathrm{nuc}})}{1 - X_n(t_{\mathrm{nuc}})} \approx \frac{\frac{1}{2} \cdot 0.11}{1 - 0.11} \approx \frac{1}{16}$$

Theoretical predictions