Seminar 2022 12 06

Chapter 2 - Inflation, continued

2.3 The Physics of Inflation

Let us remember that last week we related the requirement of an accelerated expansion, $\ddot{a} > 0$, to the requirement that

$$\varepsilon = -\frac{\dot{H}}{H^2} < 1 \quad [\text{slow-roll parameter}]$$

This can be written in an alternative way by defining

$$dN = d\ln a = H dt$$

as

$$\varepsilon = -\frac{d\ln H}{dN} < 1$$

The condition for inflation means that the relative change in H, $\frac{\Delta H}{H}$ is small per ΔN e-foldings of expansion.

To solve the horizon problem, inflation should last, thus ε should remain small, for a large number of Hubble times.

This is measured by a second slow-roll parameter

$$\kappa = \frac{\dot{\varepsilon}}{H\varepsilon} = \frac{d\ln\varepsilon}{dN} \quad [\text{this is denoted by } \eta \text{ in the lecture notes}]$$

For $|\kappa| < 1$, ε changes only slightly with changing a and inflation persists.

We want to see what microscopic physics can lead to the conditions

$$\varepsilon < 1, \quad |\kappa| < 1.$$

2.3.1 Scalar field dynamics

Let us consider a scalar field $\phi(t, \vec{x})$, which we will call the inflaton.
This field will have kinetic energy density and potential energy density $V(\phi)$
The stress-energy tensor of the scalar field is

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi - V(\phi) \right)$$

We want to consider an FRW universe dominated by such a field.
Due to the spatial isotropy and homogeneity, the scalar field should not depend on the spatial coordinates, $\partial_i \phi = 0$.

Remembering the stress-energy tensor of the perfect fluid, we have the energy density of the scalar field

$$\rho_\phi = T_{00} = \partial_0 \phi \, \partial_0 \phi - g_{00} \left(\frac{1}{2} g^{00} \partial_0 \phi \, \partial_0 \phi - V(\phi) \right) = \\ = \frac{1}{2} \dot{\phi}^2 + V(\phi)$$

On the other hand, from the spatial components, $T_{ij} = -g_{ij} P_\phi$, we have

$$T_{ij} = -g_{ij} \left(\frac{1}{2} \partial_0 \phi \, \partial_0 \phi - V(\phi) \right) \implies P_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi)$$

A condition for inflation is $P_\phi < -\frac{1}{3} \rho_\phi$. This can occur if the potential energy density dominates over the kinetic energy density,

We want an equation of motion for ϕ. Starting from the continuity equation

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + P) = 0 \implies \dot{\phi} \ddot{\phi} + \frac{dV}{d\phi} \dot{\phi} + 3H \dot{\phi}^2 = 0$$

we obtain

$$\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0$$

For $H \neq 0$ we get a friction-like term.

2.3.2 Slow roll Inflation

Let us remember the slow-roll parameter

$$\varepsilon = -\frac{\dot H}{H^2}$$

From the Friedmann equation

$$H^2 = \frac{\rho}{3M_p^2} \quad \Rightarrow \quad 2H\dot H = \frac{\dot\rho}{3M_p^2} \quad \Rightarrow \quad \dot H = \frac{\dot\rho}{6M_p^2 H} \quad \Rightarrow \quad \varepsilon = -\frac{\dot\rho}{6M_p^2H^3}$$

From the continuity equation $\dot\rho = -3H(\rho + P)$, thus

$$\varepsilon = \frac{\dot\phi^2}{2M_p^2H^2} = \frac{\frac{1}{2}\dot\phi^2}{M_p^2H^2}$$

As discussed before, $\varepsilon < 1$ implies that $\frac{1}{2}\dot\phi^2 \Big/ \frac{\rho_\phi}{3} < 1 \Rightarrow \frac{1}{2}\dot\phi^2 < \frac{\rho_\phi}{3}$

According to the lecture notes, this situation is called slow-roll inflation.

It is now said that it is useful to introduce another parameter, the dimensionless acceleration per Hubble time:

$$\delta = -\frac{\ddot\phi}{H\dot\phi}$$

Now we want to calculate the second slow-roll parameter,

$$\kappa = \frac{\dot\varepsilon}{H\varepsilon} = \frac{1}{H}\frac{d}{dt}\ln\varepsilon = \frac{1}{H}\frac{d}{dt}\left(\ln\frac{1}{2} + 2\ln\dot\phi - \ln M_p^2 - 2\ln H\right) = \frac{1}{H}\left(2\frac{\ddot\phi}{\dot\phi} - 2\frac{\dot H}{H}\right) =$$

$$= 2(\varepsilon - \delta)$$

Thus, if $\varepsilon, |\delta| \ll 1$ then $\varepsilon, |\kappa| \ll 1$.

If these conditions are satisfied, inflation does occur and persist.

Slow-roll approximation

Rewriting

$$H^2 = \frac{1}{3M_p^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right) \Rightarrow 1 = \frac{1}{3}\left(\underbrace{\frac{\frac{1}{2}\dot{\phi}^2}{M_p^2H^2}}_{\varepsilon \ll 1} + \frac{V}{M_p^2H^2}\right)$$

$$\Rightarrow H^2 \simeq \frac{V(\phi)}{3M_p^2} \qquad (*)$$

Also

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \Rightarrow \underbrace{\frac{\ddot{\phi}}{H\dot{\phi}}}_{-\delta} + 3 + \frac{1}{H\dot{\phi}}\frac{dV}{d\phi} = 0$$

$$\Rightarrow 3H\dot{\phi} \approx -\frac{dV}{d\phi} \qquad (**)$$

Combining $(*)$ and $(**)$ with previously obtained expression for ε we get

$$\varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_p^2H^2} \approx \frac{1}{2M_p^2H^2}\,\frac{1}{9H^2}\left(\frac{dV}{d\phi}\right)^2 = \frac{1\cdot 3M_p^2\cdot 3M_p^2}{2M_p^2\cdot V\cdot 9\cdot V}\left(\frac{dV}{d\phi}\right)^2 = \underbrace{\frac{M_p^2}{2}\left(\frac{\frac{dV}{d\phi}}{V}\right)^2}_{\epsilon_V}$$

Next, taking $\frac{d}{dt}$ of $(**)$ we get

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -\frac{d^2V}{d\phi^2}\dot{\phi} \quad \Big| \frac{1}{H^2\dot{\phi}}$$

$$\Rightarrow \underbrace{3\frac{\dot{H}}{H^2}}_{-3\varepsilon} + \underbrace{3\frac{\ddot{\phi}}{H\dot{\phi}}}_{-3\delta} = -\frac{1}{H^2}\frac{d^2V}{d\phi^2}$$

$$\Rightarrow -3(\varepsilon + \delta) \approx -\frac{3M_p^2}{V}\frac{d^2V}{d\phi^2} \Rightarrow \varepsilon + \delta \simeq \underbrace{\frac{M_p^2}{V}\frac{d^2V}{d\phi^2}}_{\eta_V}$$

Here we introduce the potential slow-roll parameters ϵ_V and η_V.

During slow-roll $\epsilon_V \approx \varepsilon$ and $\kappa = 2(\varepsilon - \delta) \Rightarrow \frac{\kappa}{2} = \varepsilon - \delta \Rightarrow \delta = \varepsilon - \frac{\kappa}{2}$

$$\Rightarrow \eta_V \approx 2\varepsilon - \frac{\kappa}{2}$$

Slow roll inflation occurs when these parameters are small.

Amount of inflation

Let us assume that inflation starts at t_s and ends at t_E.

These times are defined by $\varepsilon(t_s) = 1$ and $\varepsilon(t_E) = 1$.

The amount of inflation can be quanticized by the number of e-folds in the increasing scale factor:

$$N_{tot} = \int_{a_s}^{a_E} d\ln a = \int_{a_s}^{a_E} \frac{da}{a} = \int_{t_s}^{t_E} H(t)\,dt.$$

In the slow-roll regime, we can rewrite this as

$$\int_{t_s}^{t_E} H\,dt = \int_{\phi_s}^{\phi_E} \frac{H}{\dot\phi}\,d\phi = \left[\varepsilon = \frac{\frac{1}{2}\dot\phi^2}{M_p^2 H^2} \Rightarrow \frac{H}{\dot\phi} = \pm\frac{1}{\sqrt{2\varepsilon}M_p}\right] = \int_{\phi_s}^{\phi_E} \frac{1}{\sqrt{2\varepsilon}M_p}\,d\phi \approx$$

$$\approx \int_{\phi_s}^{\phi_E} \frac{1}{\sqrt{2\epsilon_V}M_p}|d\phi|$$

Here the modulus is somewhat tricky. For monotonous potentials everything should be straightforward.

It is said that to solve the horizon problem we need $N_{tot} \gtrsim 60$

Example: $V(\phi) = \frac{1}{2}m^2\phi^2$ potential.

We have $\frac{dV}{d\phi} = m^2\phi$ and $\frac{d^2V}{d\phi^2} = m^2$. Thus

$$\epsilon_V = \frac{M_p^2}{2}\frac{4}{\phi^2} = 2\left(\frac{M_p}{\phi}\right)^2, \quad \eta_V = M_p^2\frac{2}{\phi^2} = 2\left(\frac{M_p}{\phi}\right)^2$$

For inflation we need $\epsilon_V < 1$ or $2\frac{M_p^2}{\underset{>\phi_E}{\phi^2}} < 1 \Rightarrow |\phi| > \sqrt{2}M_p$

If we take ϕ_E as $\sqrt{2}M_p$ and ϕ_s as free parameter, then

$$N(\phi_s) = -\int_{\phi_s}^{\phi_E} \frac{d\phi}{M_p}\frac{1}{2\frac{M_p}{\phi}} = \frac{1}{2M_p^2}\frac{\phi^2}{2}\Big|_{\phi_E}^{\phi_s} = \frac{1}{4M_p^2}\left(\phi_s^2 - 2M_p^2\right) = \frac{\phi_s^2}{4M_p^2} - \frac{1}{2}$$

To solve the horizon problem we therefore require

$$\frac{\phi_s^2}{4M_p^2} - \frac{1}{2} > 60 \Rightarrow \phi_s^2 \gtrsim 240\,M_p^2 \Rightarrow \phi_s \gtrsim 16\,M_p$$

2.3.3 Reheating

Now we discuss what happens after inflation

Let us assume that after inflation the field ϕ starts to oscillate near the potential minimum, where we can approximate the potential as $V(\phi) = \frac{1}{2} m^2 \phi^2$.
The equation of motion then is

$$\ddot{\phi} + 3H\dot{\phi} = -m^2\phi$$

Assuming slowly expanding universe, at some point H would become relatively small. Then we could neglect the friction term.
We have the continuity equation,

$$\dot{\rho} = -3H(\rho + P) \Rightarrow \dot{\rho} + 3H\rho = -3H\underbrace{P}_{\frac{1}{2}\dot{\phi}^2 - V(\phi)} = -3H \cdot \frac{1}{2}(\dot{\phi}^2 - m^2\phi^2)$$

In the harmonic approximation, the RHS vanishes averaged over one oscillation period. Thus the inflaton field behaves as pressureless matter with $\rho_\phi \propto a^{-3}$.

To obtain matter in the universe, the inflaton field must couple with SM fields.
If inflaton can only decay into fermions, its decay should be slow:

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi$$

If the inflaton field can decay into bosons, then the decay could be very rapid due to Bose condensation effects. Then we would have „preheating", as bosons would be created far from thermal equilibrium.

In any case, the created particles should eventually thermalize.
The reheating temperature should be determined by the energy density ρ_{rh}.
After thermalization of the baryons, photons and neutrinos,
the standard Big Bang era begins.