Seminar 2022 11 29

Chapter 2 - Inflation

Initial notes

What is "The Big Bang"?
I would assume that is the singularity $a \to 0$ at $t \to 0$.
So, the Universe would be a mathematical point, and all the positions and velocities of the particles are undefined?
Yet in his book the author claims that "To specify the initial conditions of the hot Big Bang, we define the positions and velocities of all particles on an initial time slice....".

In the lecture notes the author also claims that "Notice that the Big Bang singularity is a moment in time but not a point in space. Indeed, in ... [some figures] we describe the singularity by an extended (possibly infinite) [!!!] spacelike hypersurface."

Somewhat confusing!

— — — —

In the original Inflationary universe paper by Alan Guth, there is an illuminating discussion on the initial conditions. It boils down to specifying them at some very large, but finite temperature, which should be reasonably below the Planck mass:

$$M_P = \sqrt{\frac{\hbar c}{8\pi G}} \approx 2.4 \cdot 10^{18}\ \mathrm{GeV}$$

In terms of this the Friedmann equation is given as $H^2 = \rho/(3M_P^2)$

Problems that inflation is supposed to address

1) Horizon problem
2) Flatness problem
3) Superhorizon correlations
4) Absence of magnetic monopoles

— — — —

Some comments on 4)
Various Grand Unified Theories (that say that at very high energies the electromagnetic, weak and strong forces become a single force) predict an abundance of magnetic monopoles
None are observed.
Inflation somehow reduces the expected number.

— — — —

Now let us turn to the horizon problem.
We will first need to discuss the propagation of photons (light), because this will help us to visualize the causally connected regions.
Using one parametrization of the FRW metric

$$ds^2 = a^2(\eta)\left[d\eta^2 - \left(d\chi^2 + S_k^2(\chi)\, d\Omega^2\right)\right]$$

we can orient the coordinate system so the light travels along the radial coordinate $\left(\theta = \frac{\pi}{2}, \varphi = 0\right)$:

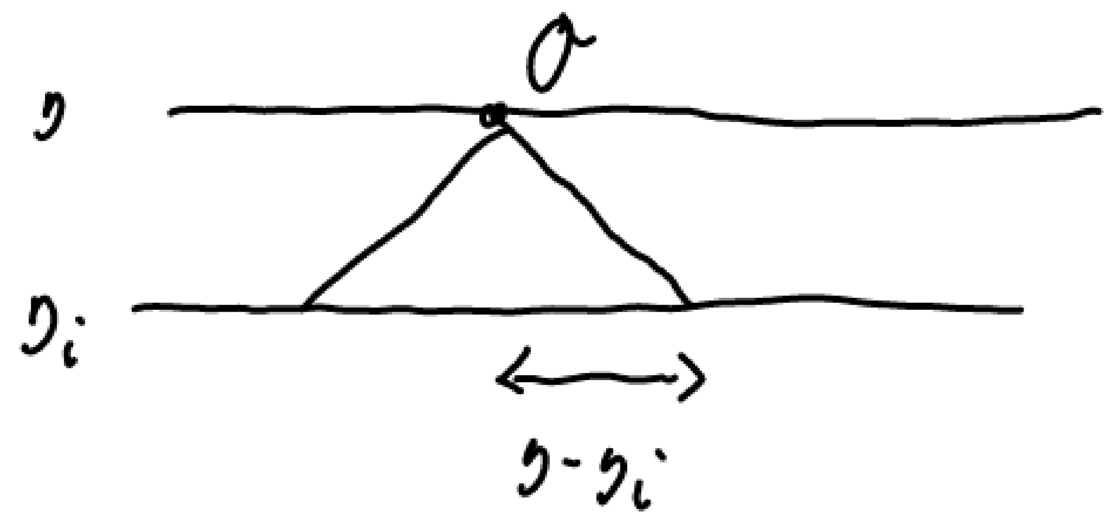
$$ds^2 = a^2(\eta)\left[d\eta^2 - d\chi^2\right]$$

For photons we have $ds^2 = 0$, thus their path is defined by

$$\Delta\chi(\eta) = \pm\Delta\eta$$

In spacetime diagrams we will therefore have 45° angles for light rays in χ-η coordinates.

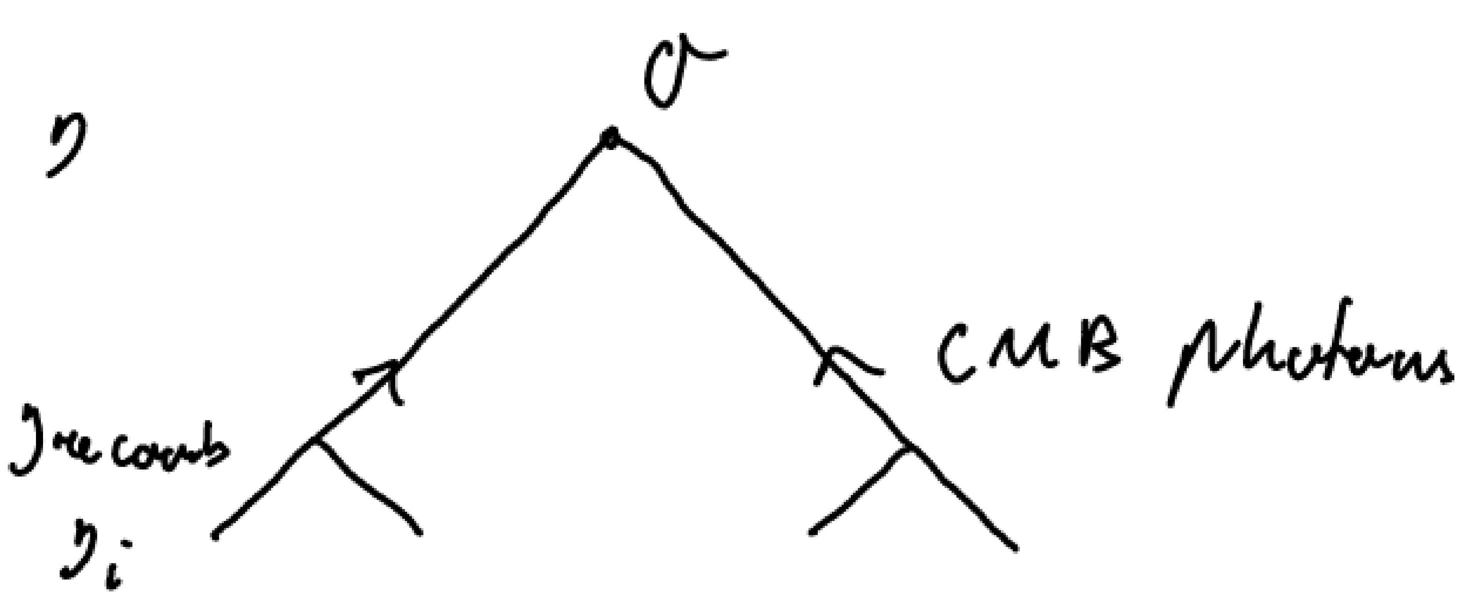
Drawing a diagram



The comoving particle horizon is

$$\chi_p(\eta) = \eta - \eta_i = \int_{t_i}^{t} \frac{dt'}{a(t')} \qquad (1)$$

Now comes illustration of the horizon problem 2.1.3



Recombination: when $p^+ + e^- \to H + \gamma$ and reverse reactions become energetically not favorable; $t_{recomb} = 380000$ years

It is said that any two points in CMB separated by more than 1° in the sky should be causally disconnected. [Tricky]

2.1.2 Hubble radius

We can rewrite (1) as

$$\chi_p = \int_{t_i}^{t} \frac{dt'}{a(t')} = \int_{a_i}^{a} \frac{da'}{a'\dot{a}'} = \int_{\ln a_i}^{\ln a} (a'H')^{-1}\, d\ln a'$$

Here

$$\dot{a} = aH$$

If we consider a universe filled with perfect fluid, for whic $P = w\rho$, we will have (neglecting the issue of sign)

$$\frac{1}{aH} = \frac{1}{\dot{a}} = \frac{1}{H_0}\, a^{\frac{1}{2}(1+3w)}$$

All normal matter sources satisfy the so-called strong energy condition, $1+3w>0$. In such a case it could be assumed that the comoving Hubble radius

Now we have to do a little bit of math:

$$\eta_p(\eta) = \int_{\ln a_i}^{\ln a} (a'H')^{-1} \, d\ln a' = \int_{\ln a_i}^{\ln a} H_0^{-1} a'^{\frac{1}{2}(1+3w)} \, d\ln a' =$$

$$= \int_{a_i}^{a} H_0^{-1} a'^{\frac{1}{2}(1+3w)-1} \, da' = H_0^{-1} \left. \frac{a'^{\frac{1}{2}(1+3w)}}{\frac{1}{2}(1+3w)} \right|_{a_i}^{a} =$$

$$= \frac{2H_0^{-1}}{(1+3w)} \left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)} \right] = \eta - \eta_i$$

If we define

$$\eta_i = \frac{2H_0^{-1}}{1+3w} a_i^{\frac{1}{2}(1+3w)}$$

and let $a_i \to 0$ (with $w > -\frac{1}{3}$), we get

$$\eta_p(\eta) = \frac{2H_0^{-1}}{1+3w} a^{\frac{1}{2}(1+3w)} = \frac{2}{1+3w} (aH)^{-1}$$

The particle horizon is thus dominated by the upper limit in the integral.

— — — —

Now some comments on the flatness problem, which is related to Ex. B.1.3

$$\Omega = \frac{\Omega_0}{\Omega_0 + (1-\Omega_0) a^{1+3w}}$$

Cases

$w = 0 \Rightarrow \Omega = \frac{\Omega_0}{\Omega_0 + (1-\Omega_0)a}$, decreases to 0 with increasing a

$w = 1/3 \Rightarrow \Omega = \frac{\Omega_0}{\Omega_0 + (1-\Omega_0)a^2}$ — " —

$w = -1 \Rightarrow \Omega = \frac{\Omega_0}{\Omega_0 + (1-\Omega_0)a^{-2}}$, goes to 1 for large a.

— — — —

Regarding the superhorizon correlations, we observe structure in CMB with larger length scale than the supposed particle horizon.

2.2 A shrinking Hubble Sphere

The lecture notes now postulate that if the Hubble radius decreased for some time during the early universe,

$$\frac{d}{dt}(aH)^{-1} < 0$$

It is not obvious why this should be a condition for inflation

This condition implies that

$$\frac{d}{dt}\frac{1}{\dot{a}} = -\frac{\ddot{a}}{\dot{a}^2} < 0 \quad \Rightarrow \quad \ddot{a} > 0 \ (\text{accelerated expansion})$$

Remembering the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(1+3w)$$

we see that acceleration is possible only for matter sources, for which

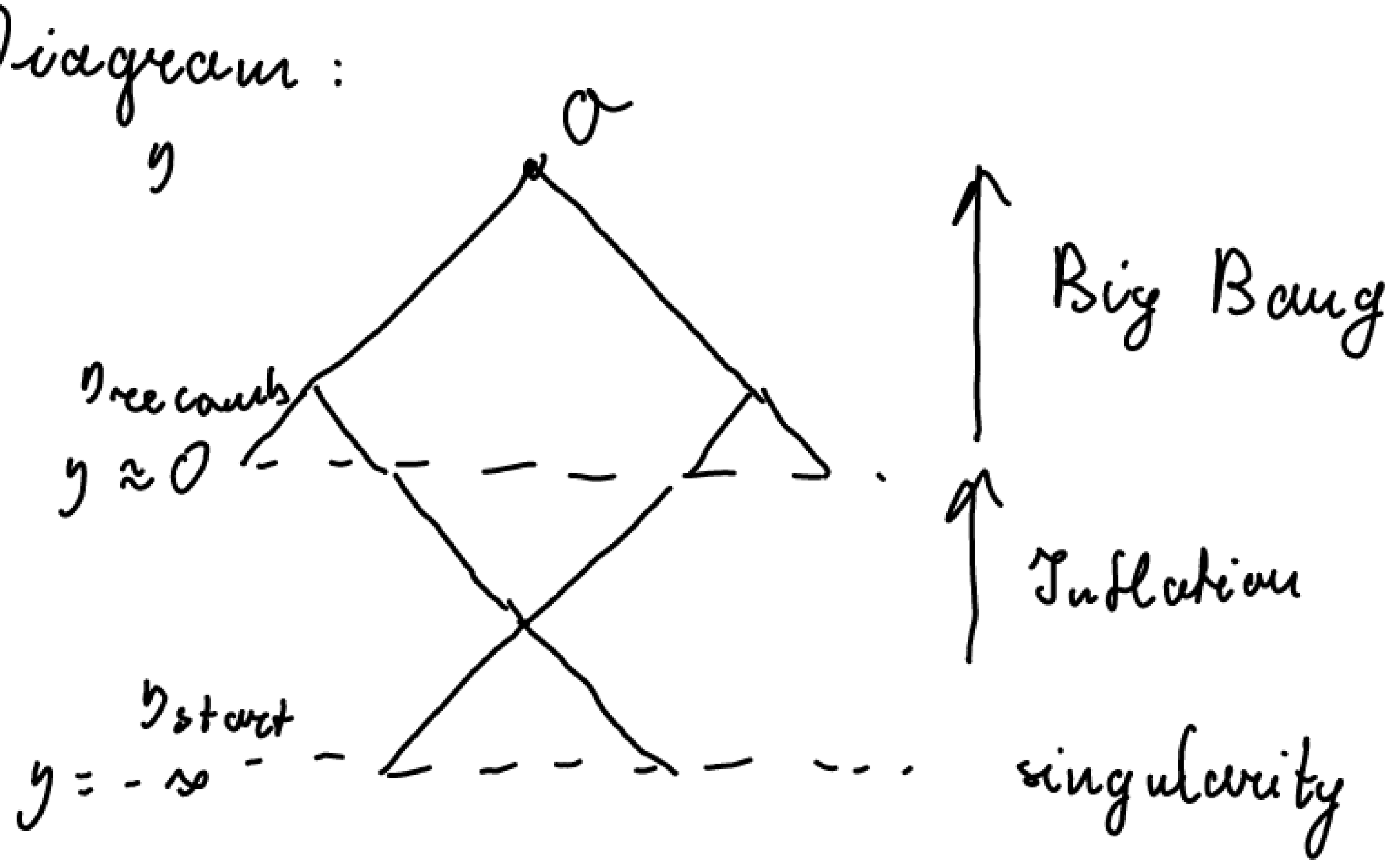
$$(1+3w) < 0$$

that is, strong energy condition is violated.

If, however, $(1+3w) < 0$, then we would have $\eta_i \to -\infty$ in calculation of the particle horizon.

This means that there would be much more conformal time in the history of the universe, thus all regions would have enough time to form causally connected space.

Diagram:



2.2.2 [Alternative] Conditions for Inflation

Following the lecture notes we discuss other ways of defining inflation.

- Slowly varying Hubble parameter

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{\dot{a}^2}{a\dot{a}^2} - \frac{a\dot{H}}{a^2H^2} = -\frac{1}{a}\left(1-\left(-\frac{\dot{H}}{H^2}\right)\right)$$

We introduce the slow roll parameter $\varepsilon = -\frac{\dot{H}}{H^2}$ to write

$$\frac{d}{dt}(aH)^{-1} = -\frac{1}{a}(1-\varepsilon)$$

Condition $\frac{d}{dt}(aH)^{-1} < 0$ is equivalent to $\varepsilon < 1$

In the limit $\varepsilon \to 0$ we would have $\dot{H} = 0$, $H = \text{const}$

$$\Rightarrow \frac{\dot{a}}{a} = \text{const} \Rightarrow \frac{da}{a} = C\,dt \Rightarrow \ln a = Ct + \ln A \Rightarrow a = Ae^{Ct}$$

According to the lecture notes, this should somehow result in

$$ds^2 = dt^2 - e^{2Ht} d\vec{x}^2$$

(de Sitter space) but I do not see how.

[Apparently, inflation is referred to as quasi de-Sitter period]

- Negative pressure

Inflation requires $1+3w < 0 \Rightarrow w < -\frac{1}{3} \Rightarrow P < 0$

- Constant energy density

The Friedmann equation $H^2 = \frac{\rho}{3M_p^2} \Rightarrow 2H\dot{H} = \frac{\dot{\rho}}{3M_p^2} \Rightarrow \dot{\rho} = 6H\dot{H}M_p^2$

$$\frac{d\ln\rho}{d\ln a} = \frac{a}{\rho}\frac{d\rho}{da} = \frac{a}{\rho}\frac{\dot{\rho}}{\dot{a}} = \frac{1}{H}\frac{\dot{\rho}}{\rho} = \frac{1}{H}\frac{6M_p^2H\dot{H}}{\rho} = \frac{6M_p^2\dot{H}}{\rho} = 2\frac{\dot{H}}{H^2} = -2\varepsilon$$

$$\Rightarrow \left|\frac{d\ln\rho}{d\ln a}\right| = 2\varepsilon$$

For small ε we would have almost constant ρ.