Problems

6.8. Dynamic response under a travelling charge. Consider the dynamical response in plasmas in the case of a travelling charge as discussed in Section 6.2.5. A direct calculation results in an undefined integral. The origin of the problem lies in the consideration that the charge was travelling from the very beginning, and hence the unperturbed state was not properly defined. The solution consists in assuming that, for infinitely negative time, the plasma was in equilibrium and the charge was turned on gradually. This is achieved by writing $\rho_{\rm ext}(\vec r, t) = \rho_0(\vec r - \vec v t)e^{\delta t}$, where δ is a small positive quantity that we will later consider to be vanishingly small. The divergence for large positive times is not of concern, because by causality it can only affect the future and not the observation time where the exponential factor remains finite (of order one in the limit $\delta \to 0^+$). Perform the Fourier transform to show that now $\omega = \vec k\cdot\vec v + i\delta$ and that the integral remains finite. Using the Plemelj formula evaluate the imaginary part of the dielectric function. Interpret its sign.

Fourier transform:

$$\widetilde h(\vec k, \omega) = (2\pi)^{-4}\int h(\vec r)e^{-i(\vec k\cdot\vec r - \omega t)}d^3\vec r\, d\vec t \tag{A.5}$$

with

$$h(\vec r, t) = \int \widetilde h(\vec k, \omega)e^{i(\vec k\cdot\vec r - \omega t)}d^3\vec r\, d\omega \ . \tag{A.6}$$

Then the immediate answer to the first part of the question would be

$$\begin{aligned}\widetilde\rho_{\rm ext}(\vec k, \omega) := \widetilde\rho_0(\vec k)\delta(\omega - \vec k\cdot\vec v) &= (2\pi)^{-4}\int \rho_0(\vec r - \vec v t)e^{\delta t}e^{i\omega t - i\vec k\cdot\vec r}d^3\vec r\, d\vec t \\ &= (2\pi)^{-4}\int \rho_0(\vec r - \vec v t)e^{i(\omega - i\delta)t - i\vec k\cdot\vec r}d^3\vec r\, d\vec t \ ,\end{aligned} \tag{1}$$

modifying $\omega \to \omega - i\delta = \omega'$.

Going through the full calculation, we starteded with the ansatz

$$f(\vec r, \vec c, t) = f_0(\vec c) + \lambda g(\vec r, \vec c, t) \ , \tag{6.23}$$

the mean field potential $\Phi_{\rm mf}$ in terms of the total charge density

$$\vec\nabla^2\Phi_{\rm mf} = -\frac{1}{\epsilon_0}\left[\rho_{\rm ext}(\vec r, t) - e\int g(\vec r, \vec c, t)d^3\vec c\right] \ , \tag{6.24}$$

and the Vlasov equation, linear in λ

$$\frac{\partial g}{\partial t} + \vec c\cdot\frac{\partial g}{\partial\vec r} + \frac{e}{m}\vec\nabla\Phi_{\rm mf}\cdot\frac{\partial f_0}{\partial\vec c} = 0 \ . \tag{6.25}$$

Fourier transforming with the modification leads to

$$-\vec k^2\widetilde\Phi_{\rm mf} = -\frac{1}{\epsilon_0}\left[\widetilde\rho_{\rm ext} - e\int\widetilde g d^3\vec c\right] \ , \tag{6.26}$$

and

$$-i\omega'\widetilde g + i\vec k\cdot\vec c\,\widetilde g + \frac{ie}{m}\widetilde\Phi_{\rm mf}\,\vec k\cdot\frac{\partial f_0}{\partial\vec c} = 0 \ . \tag{6.27}$$

The last equation gives

$$\widetilde g = -\frac{e}{m}\widetilde\Phi_{\rm mf}\frac{\vec k\cdot\frac{\partial f_0}{\partial\vec c}}{\vec k\cdot\vec c - \omega'} = -\frac{e}{m}\widetilde\Phi_{\rm mf}\frac{\vec k\cdot\frac{\partial f_0}{\partial\vec c}}{\vec k\cdot\vec c - \omega + i\delta} \ , \tag{2}$$

which would be singular for $\omega \to \vec k\cdot\vec c$, but is fine for ω'.

Next we get

$$\widetilde{\Phi}_{\rm mf} = \frac{\widetilde{\rho}_{\rm ext}(\vec{k},\omega)}{\epsilon(\vec{k},\omega)} \quad , \tag{6.28}$$

with the dielectric function

$$\epsilon(\vec{k},\omega) = \epsilon_0 \left[1 - \frac{e^2}{m\epsilon_0 k^2}\int \frac{\vec{k}\cdot\frac{\partial f_0}{\partial \vec{c}}}{\vec{k}\cdot\vec{c}-\omega+i\delta} d^3\vec{c}\right] \quad . \tag{6.30}$$

For thermal plasma we now write explicitely

$$f_0 = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-m\vec{c}^2/(2k_B T)} \quad , \tag{3}$$

giving

$$\vec{k}\cdot\frac{\partial f_0}{\partial \vec{c}} = -\left(\frac{m}{2\pi k_B T}\right)^{3/2} \frac{m\vec{k}\cdot\vec{c}}{k_B T} e^{-m\vec{c}^2/(2k_B T)} \quad , \tag{4}$$

and

$$\epsilon(\vec{k},\omega) = \epsilon_0 \left[1 + \frac{e^2}{m\epsilon_0 k^2}\frac{m}{k_B T}\left(\frac{m}{2\pi k_B T}\right)^{3/2}\int \frac{\vec{k}\cdot\vec{c}}{\vec{k}\cdot\vec{c}-\omega+i\delta} e^{-m\vec{c}^2/(2k_B T)} d^3\vec{c}\right] \quad . \tag{5}$$

Taking the coordinate system for the integration aligned to the vector $\vec{k}$, we have $\vec{c} = (c, \vec{c}_\perp)$ and $\vec{k}\cdot\vec{c} = kc$, and

$$\begin{aligned}
(\tfrac{m}{2\pi k_B T})^{3/2}\int \frac{\vec{k}\cdot\vec{c}\, e^{-\frac{m\vec{c}^2}{2k_B T}}}{\vec{k}\cdot\vec{c}-\omega+i\delta} d^3\vec{c} &= (\tfrac{m}{2\pi k_B T})^{2/2}\int e^{-\frac{m\vec{c}_\perp^2}{2k_B T}} d^2\vec{c}_\perp \left(\tfrac{m}{2\pi k_B T}\right)^{1/2}\int \frac{ck\, e^{-\frac{mc^2}{2k_B T}}}{ck-\omega+i\delta} dc \\
&= \sqrt{\tfrac{m}{2\pi k_B T}}\frac{1}{k}\int_{-\infty}^{\infty} \frac{z\, e^{-\frac{mz^2}{2k_B T k^2}}}{z-\omega+i\delta} dz = \sqrt{\tfrac{m}{2\pi k_B T}}\frac{1}{k}\int_{-\infty}^{\infty} e^{-\frac{mz^2}{2k_B T k^2}}\left(1 + \frac{\omega - i\delta}{z-\omega+i\delta}\right) dz \\
&= 1 + \sqrt{\tfrac{m}{2\pi k_B T}}\frac{\omega - i\delta}{k}\int_{-\infty}^{\infty} e^{-\frac{mz^2}{2k_B T k^2}} \frac{dz}{z-\omega+i\delta} \\
&= 1 + \sqrt{\tfrac{m}{2\pi k_B T}}\frac{\omega - i\delta}{k}\int_{-\infty}^{\infty} e^{-\frac{m(z'+\omega)^2}{2k_B T k^2}} \frac{dz'}{z'+i\delta} \quad .
\end{aligned} \tag{6}$$

The Sokhotski-Plemelj theorem states:

$$\lim_{\epsilon\to 0^+}\int_{a<0}^{b>0} \frac{f(x)}{x \pm i\epsilon} dx = \mp i\pi f(0) + \mathcal{P}\int_{a<0}^{b>0} \frac{f(x)}{x} dx \quad . \tag{7}$$

Using it on our function we get

$$\begin{aligned}
\epsilon(\vec{k},\omega) &= \lim_{\delta\to 0^+} \epsilon_0 \left[1 + \frac{e^2}{\epsilon_0 k^2 k_B T}\left(1 + \sqrt{\tfrac{m}{2\pi k_B T}}\frac{\omega - i\delta}{k}[-i\pi e^{-\frac{m\omega^2}{2k_B T k^2}} + \mathcal{P}\int_{-\infty}^{\infty} e^{-\frac{m(z'+\omega)^2}{2k_B T k^2}}\frac{dz'}{z'}]\right)\right] \\
&= \epsilon_0 \left[1 + \frac{e^2}{\epsilon_0 k^2 k_B T}\left(1 + \sqrt{\tfrac{m}{2\pi k_B T}}\frac{\omega}{k}[-i\pi e^{-\frac{m\omega^2}{2k_B T k^2}} + \mathcal{P}\int_{-\infty}^{\infty} e^{-\frac{m(z'+\omega)^2}{2k_B T k^2}}\frac{dz'}{z'}]\right)\right] \quad .
\end{aligned} \tag{8}$$

The principal value only changes the real part, so I omit it from the further discussion.

The imaginary part is

$$\mathcal{I}\mathrm{m}[\epsilon(\vec{k},\omega)] = \epsilon_0 \frac{e^2}{\epsilon_0 k^2 k_B T}\sqrt{\tfrac{m}{2\pi k_B T}}\frac{\omega}{k}(-\pi) e^{-\frac{m\omega^2}{2k_B T k^2}} < 0 \quad . \tag{9}$$

$\Rightarrow$ gain term ? (according to Wikipedia)