Problems

6.18. Conservations in the point vortex gas model. The Hamiltonian equations (6.91) conserve energy H, the angular momentum $L = \sum_i \gamma_i r_i^2$ and impulse $\vec{P} = \sum_i \gamma_i \vec{r}_i$. Write the conserved quantities in terms of the coarse-grained vorticity $\bar{\omega}$ and show that the mean-field equations respect the conservations.

Coarse graining:

$$\bar{\omega}(x,y) = \int_{\Delta x, \Delta y} \omega(\vec{r}) d^2\vec{r} = \int_{\Delta x, \Delta y} \sum_j \gamma_j \delta(\vec{r} - \vec{r}_j) d^2\vec{r} \tag{1}$$

Then we can write

$$\overline{\vec{P}} = \overline{\sum_i \gamma_i \vec{r}_i} = \int_{\Delta x, \Delta y} \vec{r} \overline{\sum_i \gamma_i \delta(\vec{r} - \vec{r}_j)} d^2\vec{r} = \int_{\Delta x, \Delta y} \vec{r}\omega(\vec{r}) d^2\vec{r} = \overline{\vec{r}}\,\bar{\omega}(\vec{r}) \tag{2}$$

$$\overline{L} = \overline{\sum_i \gamma_i \vec{r}_i^{\,2}} = \int_{\Delta x, \Delta y} \vec{r}^{\,2} \overline{\sum_i \gamma_i \delta(\vec{r} - \vec{r}_j)} d^2\vec{r} = \int_{\Delta x, \Delta y} \vec{r}^{\,2}\omega(\vec{r}) d^2\vec{r} = \overline{\vec{r}}^{\,2}\,\bar{\omega}(\vec{r}) \tag{3}$$

For the mean field quantities, I would assume:

$$\overline{\vec{v}} = \overline{\frac{\partial \vec{r}}{\partial t}} = \frac{\partial}{\partial t}\overline{\vec{r}} = \frac{d}{dt}\overline{\vec{r}} = 0 \ , \tag{4}$$

as $\vec{v}$ is assumed to be a dynamic quantity, and

$$\frac{d}{dt}\overline{\vec{P}} = \frac{d}{dt}[\overline{\vec{r}}\,\bar{\omega}] = (\frac{d}{dt}\overline{\vec{r}})\,\bar{\omega} + \overline{\vec{r}}(\frac{d}{dt}\bar{\omega}) = 0 \times \bar{\omega} + \overline{\vec{r}}(\frac{\partial}{\partial t}\bar{\omega} + \vec{\nabla}\bar{\omega} \cdot \frac{d\overline{\vec{r}}}{dt}) = \overline{\vec{r}}(\frac{\partial}{\partial t}\bar{\omega} + \overline{\vec{v}} \cdot \vec{\nabla}\bar{\omega}) = \overline{\vec{r}} \times 0 \ , \tag{5}$$

according to (6.92). And similarly

$$\frac{d}{dt}\overline{L} = \frac{d}{dt}(\overline{\vec{r}}^{\,2}\,\bar{\omega}) = \overline{\vec{r}}^{\,2}\,(\frac{d}{dt}\bar{\omega}) = 0 \tag{6}$$